MOD p POINTS ON SHIMURA VARIETIES OF PARAHORIC LEVEL

POL VAN HOFTEN, WITH AN APPENDIX BY RONG ZHOU

Abstract. We study the $\overline{\mathbb{F}}_p$-points of the Kisin–Pappas integral models of Shimura varieties of Hodge type with parahoric level. We show that if the group is quasi-split, then every isogeny class contains the reduction of a CM point, proving a conjecture of Kisin–Madapusi-Pera–Shin. We furthermore show that the mod p isogeny classes are of the form predicted by the Langlands–Rapoport conjecture (cf. Conjecture 9.2 of [47]) if either the Shimura variety is proper or if the group at p is unramified. An important ingredient in our work is a global argument that allows us to reduce the conjecture to the case of very special parahoric level. To deal with this case, we use a result of Rong Zhou on the connected components of affine Deligne–Lusztig varieties with very special level proved in the Appendix. As a corollary to our arguments, we obtain irreducibility results for Ekedahl–Oort strata.

Contents

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1. Introduction and statement of results

1.1. **Introduction.** In [32], Langlands outlines a three-part approach to prove that the Hasse–Weil zeta functions of Shimura varieties are related to L-functions of automorphic forms. The second part is about describing the mod p points of suitable integral models of Shimura varieties, which is the central topic of this article.

A conjectural description of the mod p points of integral models of Shimura varieties was first given by Langlands in [31] and was later refined by Langlands–Rapoport and Rapoport [33, 47, 49]. Together with the test function conjecture of Haines–Kottwitz [12], which was recently proved by Haines–Richarz [13], this conjecture is the main geometrical input to the Langlands–Kottwitz method for Shimura varieties of parahoric level. To explain these conjectures, we first need to introduce some notation.

Let (G, X) be a Shimura datum of Hodge type, let p be a prime number and let $U_p \subset G(\mathbb{Q}_p)$ be a parahoric subgroup. For sufficiently small compact open subgroups $U^p \subset G(\mathbb{A}_f^p)$, there is a Shimura variety $\mathbf{Sh}_U(G, X)$ of level $U = U^p U_p$, which is a smooth quasi-projective variety defined over the reflex field E. For a prime $v|p$ of E, there should be a *canonical* integral model $\mathscr{S}_U(G, X)$ over $\mathcal{O}_{E,(v)}$.

When U_p is hyperspecial, canonical integral models should be smooth and are unique if they satisfy a certain extension property (cf. [38]). Recent work [42, 43] of Pappas and Pappas–Rapoport defines a notion of canonical integral models when U_p is an arbitrary parahoric and proves that they are unique if they exist.

Then there should be a bijection (see [33, Section 5] and [47, Conjecture 9.2])

$$\varprojlim_{U^p} \mathscr{S}_{U^p U_p}(G, X)(\overline{\mathbb{F}}_p) \simeq \coprod_{\phi} S(\phi), \tag{1.1.1}$$

where

$$S(\phi) = I_\phi(\mathbb{Q}) \backslash X_p(\phi) \times X^p(\phi).$$

Let us elaborate: The sets $S(\phi)$ are supposed to correspond to points in a single isogeny class, with $X_p(\phi)$ parametrising p-power isogenies, $X^p(\phi)$ parametrising prime-to-p isogenies and $I_\phi(\mathbb{Q})$ the group of self quasi-isogenies. The set $X^p(\phi)$ is a $G(\mathbb{A}_f^p)$-torsor and $X_p(\phi)$ is a subset of $G(\mathbb{Q}_p^{\mathrm{ur}})/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$, where $\mathcal{G}/\mathbb{Z}_p$ is the parahoric group scheme with $\mathcal{G}(\mathbb{Z}_p) = U_p$. In fact, the set $X_p(\phi)$ is the set of $\overline{\mathbb{F}}_p$-points of an affine Deligne–Lusztig variety, see Section 2.4.2.

In the unramified PEL case, (1.1.1) corresponds to Rapoport–Zink uniformisation of isogeny classes (see [46, Section 6]), with $X_p(\phi)$ corresponding to the set of $\overline{\mathbb{F}}_p$-points of a Rapoport–Zink space. This is why we will often refer to (1.1.1) as *uniformisation* of isogeny classes. Uniformisation of isogeny classes for Shimura varieties of Hodge type is often assumed in recent work in the area, cf. [14, 21, 43].

We also expect that (1.1.1) is compatible with the action of $G(\mathbb{A}_f^p)$ on both sides, and that the action of Frobenius on the left-hand side should correspond to the action of a certain operator Φ on the right-hand side. If $G_{\mathbb{Q}_p}$ is quasi-split, then we moreover expect that each isogeny class contains the reduction of a special point, see [28, Conjecture 1].

1.2. **Main results.** Let (G, X) be a Shimura datum of Hodge type and let $p > 2$ be a prime. We will assume throughout this introduction that: The group $G_{\mathbb{Q}_p}$ is quasi-split and splits over a tamely ramified extension, the prime p does not divide the order of $\pi_1(G^{\mathrm{der}})$ and $\pi_1(G)_{I_p}$ is torsion-free[1]. Here $I_p \subset \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is the inertia group and $\pi_1(G)$ is the algebraic fundamental group of G.

Let $U_p \subset G(\mathbb{Q}_p)$ be a parahoric subgroup, let $U^p \subset G(\mathbb{A}_f^p)$ be a sufficiently small compact open subgroup and consider the Shimura variety $\mathbf{Sh}_U(G, X)$ of level $U = U^p U_p$ which is an algebraic variety over the reflex field E. By [25, Theorem 0.1], this Shimura variety has an extension to a flat normal scheme $\mathscr{S}_U(G, X)$ over $\mathcal{O}_{E,(v)}$, where $v|p$ is a prime of the reflex field E. Under our assumptions, these integral models are canonical in the sense of [42, Definition 7.1.3], see [42, Theorem 1.4].

Theorem 1. *Let (G, X) be a Shimura variety of Hodge type as above. Then each isogeny class of $\mathscr{S}_U(G, X)(\overline{\mathbb{F}}_p)$ contains a point x which is the reduction of a special point on $\mathbf{Sh}_U(G, X)$.*

This confirms [28, Conjecture 1]. Theorem 1 for very special parahoric subgroups U_p is part 2 of Theorem A.4.5 of the appendix by Rong Zhou.

Theorem 1 was proved by Kisin when U_p is a hyperspecial subgroup, see [27], and proved by Zhou when $G_{\mathbb{Q}_p}$ is residually split, see [55]. We remind the reader that split implies residually split implies quasi-split, and that residually split and unramified implies split. As in [27, 55], such a lifting result is deduced from uniformisation of isogeny classes, which is our second main result. Part 1 of the next theorem is part 1 of Theorem A.4.5 of the appendix.

Theorem 2. *Let (G, X) be as above and let U_p denote a parahoric subgroup of $G(\mathbb{Q}_p)$.*

(1) If U_p is very special, then each isogeny class of $\mathscr{S}_U(G, X)(\overline{\mathbb{F}}_p)$ has the form

$$I_\phi(\mathbb{Q})\backslash X_p(\phi) \times X^p(\phi)/U^p.$$

(2) If either $G_{\mathbb{Q}_p}$ splits over an unramified extension or if $\mathbf{Sh}_U(G, X)$ is proper, then the same conclusion holds for arbitrary parahoric subgroups U_p.

As a consequence of part 2 of Theorem 2, we verify that the He–Rapoport axioms of [16] hold for the Kisin–Pappas integral models. All but one of the axioms (Axiom 4(c)) were proved in earlier work of Zhou, see [55].

Theorem 3. *Let (G, X) be a Shimura datum of Hodge type as above. If either $G_{\mathbb{Q}_p}$ splits over an unramified extension or if $\mathbf{Sh}_U(G, X)$ is proper, then the He–Rapoport axioms of* [16, Section 3] *hold for the Kisin–Pappas integral models.*

Combining our proof of part 2 of Theorem 2 with the ℓ-adic monodromy theorem of [23], we obtain an irreducibility result for the Ekedahl–Kottwitz–Oort–Rapoport (EKOR) strata defined by Shen–Yu–Zhang in [50]. We assume for simplicity that G^{ad} is simple over $\mathbb{Q}$, see Theorem 4.5.2 for a more general statement.

Theorem 4. *Let (G, X) be as above, and let U_p denote a very special parahoric. Let $\mathscr{S}_{U,\overline{\mathbb{F}}_p}\{w\}$ be an EKOR stratum that is not contained in the smallest Newton stratum. If either $G_{\mathbb{Q}_p}$ splits over an unramified extension or if $\mathbf{Sh}_U(G, X)$ is proper, then the natural map*

$$\mathscr{S}_{U,\overline{\mathbb{F}}_p}\{w\} \to \mathscr{S}_{U,\overline{\mathbb{F}}_p}(G, X)$$

induces a bijection on sets of connected components.

[1]For Shimura data of abelian type that are not of type $D^{\mathbb{H}}$ in the sense of [37, Appendix B], one can always find an auxiliary Shimura datum of Hodge type where the last two conditions are satisfied, see [25, Lemma 4.6.22].

In particular, this proves an irreducibility result for Ekedahl–Oort (EO) strata, which are the EKOR strata at hyperspecial level. Theorem 4 was proved by Ekedahl and van der Geer [8] in the Siegel case. Theorem 4 is used in [24] to prove irreducibility results for Igusa varieties.

1.3. **Overview of the proof.** Both [27] and [55] employ roughly the same strategy, which we will now briefly sketch: The integral models $\mathscr{S}_U(G,X)$ of Shimura varieties of Hodge type come equipped, by construction, with finite maps $\mathscr{S}_U(G,X) \to \mathscr{S}_{K'}(\mathrm{GSp}, S^{\pm})$ to Siegel modular varieties. Given a point $x \in \mathscr{S}_U(G,X)(\overline{\mathbb{F}}_p)$, classical Dieudonné theory produces a map

$$X_p(\phi) \to \mathscr{S}_{U'}(\mathrm{GSp}, S^{\pm})(\overline{\mathbb{F}}_p),$$

and the main difficulty is to show that it factors through $\mathscr{S}_U(G,X)$. A deformation theoretic argument shows that it suffices to prove this factorisation for one point on each connected component of $X_p(\phi)$, and therefore we need to understand these connected components. In the hyperspecial case, this is done in [6], and in the parahoric case this is done in [19], under the assumption that $G_{\mathbb{Q}_p}$ is residually split. The main obstruction to extend the methods of [55] beyond the residually split case, is that we do not understand connected components of affine Deligne–Lusztig varieties of parahoric level for more general groups.[2]

1.3.1. In Appendix A, Rong Zhou studies connected components of affine Deligne–Lusztig varieties for quasi-split groups and very special parahoric subgroups, generalising results of [6] and [39] in the case of unramified groups. In particular, part 1 of Theorem 2 and Theorem 1 in the case of a very special parahoric are proved there, see Theorem A.4.5.

1.3.2. In Section 4 we prove uniformisation for a general parahoric subgroup by reducing to the case of a very special parahoric subgroup. This reduction argument happens on the level of Shimura varieties; we will give a brief overview of our reduction argument below.

Let U_p denote a very special parahoric subgroup and let U_p' denote an Iwahori subgroup contained in U_p, then by [55, Section 7] there is a proper morphism of integral models $\mathscr{S}_{U'}(G,X) \to \mathscr{S}_U(G,X)$ and we let $\mathrm{Sh}_{U_p'} \to \mathrm{Sh}_{U_p}$ be the induced morphism on the perfections of their special fibers. There is a commutative diagram

$$\begin{array}{ccc} \mathrm{Sh}_{G,U'} & \longrightarrow & \mathrm{Sht}_{G,\mu,U_p'} \\ \downarrow & & \downarrow \\ \mathrm{Sh}_{G,U} & \longrightarrow & \mathrm{Sht}_{G,\mu,U_p}, \end{array} \tag{1.3.1}$$

where Sht_{G,μ,U_p} is the stack of parahoric U_p-shtukas of type μ introduced by Xiao-Zhu [54] (cf. Section 2.2.7, 2.2.12 and [50, Section 4]), with μ the inverse of the Hodge cocharacter induced by the Shimura datum (G,X).

The horizontal morphisms in (1.3.1) are the Hodge type analogues of the morphism from the moduli space of abelian varieties to the moduli stack of quasi-polarised Dieudonné modules. If $G = \mathrm{GSp}$, then this diagram is Cartesian. In general, it follows from 'local uniformisation' of $\mathrm{Sht}_{G,\mu,U'}$, that isogeny classes in $\mathrm{Sh}_{G,U_p'}$ have the correct form if (1.3.1) is Cartesian, see Theorem 3.3.1. One of the main technical results of this paper, Theorem 4.4.1, is that the diagram is Cartesian under the assumptions of part 2 of Theorem 2, which proves a conjecture of He and Rapoport that we learned from Rong Zhou.

[2]After a first version of our paper appeared, we learned of work of Nie [40] which solves this problem for unramified groups. Recently there has been work of Gleason–Lim–Xu [10] and Gleason–Lourenço [11] which completely settles the problem of understanding connected components of affine Deligne–Lusztig varieties.

1.3.3. We prove in Section 2, see Proposition 2.2.15, that the morphism $\mathrm{Sht}_{G,\mu,U_p'} \to \mathrm{Sht}_{G,\mu,U_p}$ is representable by perfectly proper algebraic spaces, and we let $\mathrm{Sh}_{G,U',\star}$ be the fiber product of (1.3.1). There is a closed immersion $\iota : \mathrm{Sh}_{G,U'} \to \mathrm{Sh}_{G,U',\star}$ given by the universal property of the fiber product, see Proposition 4.1.4. To prove the main theorem, it suffices to show that ι is an isomorphism.

We first show that $\mathrm{Sh}_{G,U',\star}$ is equidimensional of the same dimension as $\mathrm{Sh}_{G,U'}$ and that it has a Kottwitz–Rapoport (KR) stratification with the expected properties. To do this, we build a local model diagram for $\mathrm{Sh}_{G,U',\star}$ in the world of perfect algebraic geometry, see Proposition 4.2.1. This requires us to produce a version of the diagram in (1.3.1) for stacks of restricted shtukas, and to analyse the forgetful maps for these stacks. Another key ingredient is the fact, proved[3] by Xiao–Zhu, [54], and Shen–Yu–Zhang, [50], that the morphisms from $\mathrm{Sh}_{G,U}$ to these stacks of restricted shtukas are perfectly smooth.

The next step is to study the irreducible components of $\mathrm{Sh}_{G,U',\star}$ and $\mathrm{Sh}_{G,U'}$. In Section 4.3, see Proposition 4.3.16, we will show that each irreducible component of $\mathrm{Sh}_{G,U',\star}$ can be moved into $\mathrm{Sh}_{G,U'}$ using prime-to-p Hecke operators. Since $\mathrm{Sh}_{G,U',\star}$ is stable under the prime-to-p Hecke operators, we may conclude from this that $\iota : \mathrm{Sh}_{G,U'} \to \mathrm{Sh}_{G,U',\star}$ is an isomorphism.

To prove Proposition 4.3.16, we use the KR stratification of both $\mathrm{Sh}_{G,U'}$ and $\mathrm{Sh}_{G,U',\star}$ to reduce to analysing irreducible components in each KR stratum separately. Our proof then proceeds by degenerating to the zero-dimensional KR stratum, which we describe explicitly using Rapoport–Zink uniformisation of the basic locus.

Our assumption that either $G_{\mathbb{Q}_p}$ splits over an unramified extension or that $\mathbf{Sh}_U(G,X)$ is proper will be used to prove that every irreducible component of the closure of a KR stratum in $\mathrm{Sh}_{G,U',\star}$ intersects the zero-dimensional KR stratum, see Lemma 4.3.4 and Proposition 4.3.5. In the proper case, it is enough to prove that KR strata in $\mathrm{Sh}_{G,U',\star}$ are quasi-affine. In the unramified case, we use results of [53] and [1] on the Ekedahl–Oort stratification and results of [17] on the geometry of forgetful maps.

1.4. **Outline of the paper.** In Section 2 we will study forgetful maps for moduli stacks of local shtukas and moduli stacks of restricted local shtukas. We will also study Newton strata in moduli spaces of shtukas and describe them explicitly in terms of affine Deligne–Lusztig varieties. In Section 3 we study uniformisation of isogeny classes in Shimura varieties of Hodge type at parahoric level. We will deduce the existence of CM lifts at arbitrary parahoric level from the results of Appendix A, and we will show that uniformisation for general parahoric subgroups is equivalent to a certain diagram being Cartesian. In Section 4, we prove that this diagram is Cartesian.

2. Local shtukas

We start this section by recalling some perfect algebraic geometry from [54, Appendix A] and defining a notion of weakly perfectly smooth morphisms of perfect algebraic stacks.

In the rest of the section we will recall the moduli stacks of local shtukas with parahoric level of [50] and study the forgetful maps between them. We start by proving Proposition 2.2.15, which states that this forgetful map is representable and proper. We then study forgetful maps of restricted local shtukas and prove Proposition 2.3.4, which is an important technical result that will be used in Section 3 to prove equidimensionality of $\mathrm{Sh}_{G,U_p',\star}$.

In the second half, we discuss σ-conjugacy classes and the Newton stratification on moduli stacks of local shtukas. We end by discussing affine Deligne–Lusztig varieties and use them in Lemma 2.4.5 to describe Newton strata in moduli stacks of local shtukas. This latter result is used in Section 3 to 'lift' uniformisation along forgetful maps.

[3] See Remark 3.1.8.

2.1. **Some perfect algebraic geometry.** We will use the language of perfect algebraic geometry from [57, Appendix A]. Let $k = \overline{\mathbb{F}}_p$ and denote by $\mathbf{Aff}_k^{\mathrm{perf}}$ the category of perfect k-algebras, equipped with the étale topology. Perfect k-schemes define fpqc sheaves on $\mathbf{Aff}_k^{\mathrm{perf}}$, and for X a scheme over k we will write X^{perf} for the perfection of X (the inverse limit over the relative k-Frobenius of X).

Perfect algebraic spaces are defined to be sheaves on $\mathbf{Aff}_k^{\mathrm{perf}}$ that are étale locally isomorphic to perfect schemes. See [54, Definition A.1.7] for the definition of a pfp (perfectly of finite presentation) algebraic space. A perfect algebraic space is pfp if and only if it is isomorphic to the perfection of an algebraic space of finite presentation over k, see [54, Proposition A.1.8]. We will often write pfp algebraic space to mean pfp perfect algebraic space.

Lemma 2.1.1. *If X is a pfp algebraic space. Then for every directed set I and any inverse system $(T_i, f_{i,j})$ of perfect k-schemes with all $f_{i,j}$ affine and all T_i qcqs, the natural map*

$$\mathrm{Hom}(\varprojlim_i T_i, X) \to \varinjlim_i \mathrm{Hom}(T_i, X)$$

is a bijection.

Proof. Choose a deperfection $X \to Y$ of X using [54, Proposition A.1.8]. We may then apply [51, Proposition 01ZC] to deduce that the natural map

$$\mathrm{Hom}(\varprojlim_i T_i, Y) \to \varinjlim_i \mathrm{Hom}(T_i, Y), \tag{2.1.1}$$

is an isomorphism. We conclude by noting that (2.1.1) can be identified

$$\mathrm{Hom}(\varprojlim_i T_i, X) \to \varinjlim_i \mathrm{Hom}(T_i, X)$$

since the T_i are perfect and since T is perfect. □

2.1.2. We will use the notion of perfectly proper morphism of perfect algebraic spaces, see [57, Definition A.18]. A morphism $f : X \to Y$ of pfp algebraic spaces over k is perfectly proper if and only if it is isomorphic to the perfection of a proper morphism of algebraic spaces of finite presentation over k, see [57, Lemma A.19]. We will often write perfectly proper algebraic space to mean a perfect algebraic space whose structure map to $\operatorname{Spec} k$ is perfectly proper.

Recall that a morphism $f : X \to Y$ of perfect algebraic spaces is called *perfectly smooth of relative dimension d at x*, where $x \in X$, if there is an étale neighbourhood $U \to X$ of x and $V \to Y$ of $f(x)$ such that $U \to X \to Y$ factors through a map $h : U \to V$ and such that h factors as $h = \mathrm{pr} \circ h'$ where

$$h' : U \to (\mathbb{A}^d)^{\mathrm{perf}} \times V$$

is étale and where $\mathrm{pr} : (\mathbb{A}^d)^{\mathrm{perf}} \times V \to V$ is the projection. It is called *perfectly smooth of relative dimension d* if it is perfectly smooth of relative dimension d at all points $x \in X$. This property is preserved under base change, and the composition of a perfectly smooth morphism f of relative dimension d with a perfectly smooth morphism g of relative dimension d' is perfectly smooth of relative dimension $d + d'$. A morphism $f : X \to Y$ is called *perfectly smooth* if it is perfectly smooth of *some* dimension at every $x \in X$. This property is also preserved under base change and composition.

Example 2.1.3. If $f : X \to Y$ is a morphism of schemes over k that is smooth of relative dimension d at $x \in X$, then $f^{\mathrm{perf}} : X^{\mathrm{perf}} \to Y^{\mathrm{perf}}$ is perfectly smooth of relative dimension d at x by [51, Lemma 054L].

Lemma 2.1.4. *Let $f : X \to Y$ be a perfectly smooth morphism of perfect algebraic spaces. If X is connected, then f is perfectly smooth of relative dimension d for some integer d.*

Proof. Let $x \in X$. Then f is perfectly smooth of relative dimension d at x for some integer d, and it follows from the definition that the same is true for all u in an open neighborhood W of x. Now let $x' \in X$. Then f is perfectly smooth of relative dimension d' at x' for some integer d', and it suffices to show that $d = d'$. Note that f is perfectly smooth of relative dimension d' at u' for all u' in an open neighborhood W' of x'.

Since X is connected, we know $W' \cap W$ is non-empty and thus there is a point $u'' \in W' \cap W$ such that f is perfectly smooth of relative dimension d and d' at u''. It follows from the definition that there are open neighborhoods U and U' of u' such $f^{-1}(f(u)) \cap U$ is étale locally isomorphic to $(\mathbb{A}^d)^{\mathrm{perf}}$ and such that $f^{-1}(f(u)) \cap U'$ is étale locally isomorphic to $(\mathbb{A}^{d'})^{\mathrm{perf}}$. For dimension reasons, we must then have $d = d'$. □

Definition 2.1.5. A morphism $f : Y \to Z$ of perfect algebraic spaces is called *weakly perfectly smooth of relative dimension d at y* for $y \in Y$ if: There exists an open neighborhood U of y and a surjective map $g : X \to U$ that is perfectly smooth of relative dimension e, where X is a perfect algebraic space, such that $f \circ g : X \to Z$ is perfectly smooth of relative dimension $e + d$. A morphism is called *weakly perfectly smooth of relative dimension d* if it is weakly perfectly smooth of relative dimension d at y for all $y \in Y$.

This property is preserved under base change, and the composition of a weakly perfectly smooth morphism of relative dimension d with a weakly perfectly smooth morphism of relative dimension d' is a weakly perfectly smooth morphism of relative dimension $d + d'$. The following lemmas show that the integer d is well-defined.

Lemma 2.1.6. *Let $f : X \to Y$ be an open morphism of equidimensional pfp algebraic spaces such that the fibers of f are equidimensional of dimension d. Then* $\dim X + d = \operatorname{Dim} Y$.

Proof. This follows from the corresponding fact for open morphisms of finite type schemes over k, by choosing a deperfection of f, using [57, Proposition A.17], and applying [51, Lemma 0AFE] and its proof. □

Lemma 2.1.7. *If $f : Y \to Z$ is weakly perfectly smooth of relative dimension d at $y \in Y$, then there is an open neighborhood U of y such that $U \cap f^{-1}(f(y))$ is equidimensional of dimension d.*

Proof. The statement is Zariski local on the source, so after replacing Y by an open neighborhood of a point in Y there is a surjective map $g : X \to Y$ that is perfectly smooth of relative dimension e, such that $f \circ g : X \to Z$ is perfectly smooth of relative dimension $e + d$.

Then the scheme $(f \circ g)^{-1}(z)$ is equidimensional of dimension $e+d$ since $f \circ g$ is perfectly smooth of relative dimension $e+d$. Similarly the fibers of the fpqc cover $(f \circ g)^{-1}(z) \to f^{-1}(z)$ are equidimensional of dimension e. It now follows from Lemma 2.1.6 that $f^{-1}(z)$ is equidimensional of dimension d. □

A morphism $f : Y \to Z$ is called *weakly perfectly smooth* if there is a perfectly smooth surjection $g : X \to Y$ such that $f \circ g$ is perfectly smooth. The following lemma relates this to Definition 2.1.5.

Lemma 2.1.8. *A morphism $f : Y \to Z$ is weakly perfectly smooth if and only if for all $y \in Y$ the morphism f is weakly perfectly smooth of relative dimension d_y at y, for some positive integer d_y which is allowed to depend on y.*

Proof. If f is weakly perfectly smooth, then there is a perfectly smooth surjection $g : X \to Y$ such that $f \circ g$ is perfectly smooth. For $y \in Y$ there is a connected component V of X whose image U_y contains y, and such that $g_V := g\big|_V$ and $f \circ g_V$ are perfectly smooth. Since V is connected, the morphisms g_V and $f \circ g_V$ are perfectly smooth of relative dimensions e and $e + d$, for some e and d, see Lemma 2.1.4. Since perfectly smooth morphisms are open, we see that U_y is open and so f is weakly perfectly smooth of relative dimension d at y.

Conversely, if for all $y \in Y$ the morphism f is weakly perfectly smooth of relative dimension d_y at y, then for each $y \in Y$ there is an open subset $U_y \subset Y$ containing y and a perfectly smooth surjection $g_y : X_y \to U_y$ such that $f \circ g_y$ is perfectly smooth. Then the natural map

$$g : \coprod_y X_y \to Y$$

is perfectly smooth and surjective and similarly $f \circ g$ is also perfectly smooth. It follows that f is weakly perfectly smooth. □

Lemma 2.1.9. *Let $f : Y \to Z$ be a weakly perfectly smooth morphism of perfect algebraic spaces. If Y is connected, then f is weakly perfectly smooth of relative dimension d for some d.*

Proof. It follows from Lemma 2.1.8 that for $y \in Y$ there exists a positive integer d_y such that f is weakly perfectly smooth of relative dimension d_y at y. Moreover, the same is true for all u in an open neighborhood U_y of y.

Thus if $y, y' \in Y$ with positive integers d_y, d'_y and open neighborhoods $U_y, U_{y'}$, then $U_y \cap U_{y'}$ is non-empty because Y is connected. Therefore there is a point $u \in U_y \cap U_{y'}$ such that f is weakly perfectly smooth of relative dimensions d and d' at u. By Lemma 2.1.7, it follows that $d = d'$ and we conclude that f is weakly perfectly smooth of relative dimension d. □

Recall that a scheme is called *locally integral* if its local rings are integral.

Lemma 2.1.10. *Let $f : Y \to Z$ be a weakly perfectly smooth morphism of perfect schemes. If Z is locally integral, then Y is locally integral.*

Proof. Let $y \in Y$. Then by Lemma 2.1.8 the morphism f is weakly perfectly smooth of relative dimension d at Y. Therefore there is an open subset $U_y \subset Y$ containing y and a surjective map $g : X \to U$ that is perfectly smooth of relative dimension e, such that $f \circ g : X \to Z$ is perfectly smooth of relative dimension $e + d$. To show that the local ring of Y at y is integral, it suffices by fpqc descent to do this for the local rings of X.

Let $x \in X$ and choose an étale neighbourhood $U \to X$ of x and $V \to Z$ of $(f \circ g)(x)$ such that $U \to X \to Z$ factors through a map $h : U \to V$ and such that h factors as $h = \mathrm{pr} \circ h'$ where

$$h' : U \to (\mathbb{A}^{d+e})^{\mathrm{perf}} \times V$$

is étale and where $\mathrm{pr} : (\mathbb{A}^{d+e})^{\mathrm{perf}} \times V \to V$ is the projection.

The local rings of $(\mathbb{A}^{d+e})^{\mathrm{perf}} \times V$ are localisations of perfected polynomial rings over the local rings of V, and thus integral. Therefore the local rings of U are integral, and by étale descent the local ring of X at x is integral. □

2.1.11. We follow [51, Section 04XB] to define certain properties of morphisms of prestacks on $\mathbf{Aff}_k^{\mathrm{perf}}$ that are representable by morphisms of perfect algebraic spaces. For example, a morphism $f : X \to Y$ of prestacks that is representable by perfect algebraic spaces is called *perfectly smooth* if it is representable by perfectly smooth morphisms of perfect algebraic spaces. In other words, if for every morphism $T \to Y$, where T is a perfect algebraic space, the base change $X_T \to T$ is a perfectly smooth morphism of perfect algebraic spaces.

A *pfp algebraic stack* is a stack Y on $\mathbf{Aff}_k^{\mathrm{perf}}$ for the fpqc topology with diagonal representable by pfp algebraic spaces that admits a perfectly smooth surjective[4] map $f : U \to Y$ from a pfp algebraic space. The main example that we will be interested in is the quotient stack[5] $[X/G]$ of a pfp algebraic space X by a pfp group scheme G. This is a pfp algebraic stack because $X \to [X/G]$ is perfectly

[4]This means per definition that f is representable by perfectly smooth surjections of perfect algebraic spaces.

[5]We always take quotient stacks in the étale topology unless otherwise specified.

smooth, see Example 2.1.13. We will also need a notion of weak perfect smoothness for morphisms of pfp algebraic stacks that are not necessarily representable.

Definition 2.1.12. A morphism $f : Y \to Z$ of pfp algebraic stacks is called *weakly perfectly smooth*, if there is a perfectly smooth surjective morphism $g : X \to Y$ from a pfp algebraic space X such that the composition $f \circ g$ is perfectly smooth.

As before, this property is preserved under base change and composition. If $f : Y \to Z$ is representable, then this is (per definition) equivalent to asking that $f : Y \to Z$ is representable by weakly perfectly smooth morphisms of perfect algebraic spaces.

Example 2.1.13. Let G be a pfp group scheme over $\operatorname{Spec} k$. Then $G \to \operatorname{Spec} k$ arises as the perfection of a smooth group scheme over k by [57, Lemma A.26], and therefore $G \to \operatorname{Spec} k$ is perfectly smooth by Example 2.1.3. This means that G-torsors for the étale topology are perfectly smooth morphisms. In particular, the natural map $\operatorname{Spec} k \to [\operatorname{Spec} k/G]$ is perfectly smooth and thus $[\operatorname{Spec} k/G] \to \operatorname{Spec} k$ is weakly perfectly smooth.

Example 2.1.14. Recall that an étale G-gerbe over a pfp algebraic stack Y is a morphism $f : X \to Y$ of pfp algebraic stacks that is étale locally (on Y) of the form $Y \times [\operatorname{Spec} k/G] \to Y$. Since $[\operatorname{Spec} k/G] \to \operatorname{Spec} k$ is weakly perfectly smooth, it follows that $f : X \to Y$ is weakly perfectly smooth because this can be checked étale locally on Y.

Remark 2.1.15. In [54, Definition A.1.13], a morphism of pfp algebraic stacks satisfying the property in Definition 2.1.12 is called a perfectly smooth morphism. However, it is not clear to us that a morphism $f : Y \to Z$ of pfp algebraic spaces satisfying the property in Definition 2.1.12 is perfectly smooth (in the sense defined in the beginning of Section 2.1), rather than just weakly perfectly smooth.

Lemma 2.1.16. *Suppose that X is a pfp algebraic space that is equidimensional of dimension d with an action of a pfp group scheme G, and let Y be a pfp algebraic space together with a weakly perfectly smooth morphism*

$$f : Y \to [X/G] .$$

Then Y is equidimensional if and only if f is weakly perfectly smooth of relative dimension n, where $\operatorname{Dim} Y = d + n - \dim G$.

Proof. Consider the fiber product diagram

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{f}} & X \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & [X/G] . \end{array}$$

The morphism $\tilde{Y} \to Y$ is a G-torsor and hence perfectly smooth of relative dimension equal to $\dim G$ by Example 2.1.13. If f is weakly perfectly smooth of relative dimension n, then $\tilde{f}$ is weakly perfectly smooth of relative dimension n because it is a basechange of f. It follows from Lemma 2.1.7 and Lemma 2.1.6 that $\operatorname{Dim} \tilde{Y} = d + n$. Similarly, it follows that the dimension of Y is equal to $d + n - \dim G$.

If Y is equidimensional of dimension $\operatorname{Dim} Y$, then $\tilde{Y}$ is equidimensional of dimension $\operatorname{Dim} Y + \operatorname{Dim} G$ by Lemma 2.1.6 and Lemma 2.1.7. Since both X and $\tilde{Y}$ are equidimensional and pfp, we see that the dimension of the fibers of $\tilde{f}$ must be equal to $\operatorname{Dim} Y + \operatorname{Dim} G - d$ by Lemma 2.1.6. Therefore $\tilde{f}$ is weakly perfectly smooth of relative dimension $n = \operatorname{Dim} Y + \operatorname{Dim} G - d$, and it readily follows from this that the same holds for f. □

2.2. **Affine flag varieties, moduli stacks of shtukas and forgetful maps.**

2.2.1. Let $\breve{\mathbb{Q}}_p$ be the completion of a maximal unramified extension of $\mathbb{Q}_p$, which is equipped with an action of the p-Frobenius σ. Let G be a connected reductive group over $\mathbb{Q}_p$ and let $B(G,\mathbb{Q}_p)$ (resp. $B(G,\breve{\mathbb{Q}}_p)$) denote the (extended) Bruhat–Tits building of G over $\mathbb{Q}_p$ (resp. $\breve{\mathbb{Q}}_p$). For a non-empty bounded subset $\Xi \subset B(G,\mathbb{Q}_p)$ which is contained in an apartment, we let $G(\mathbb{Q}_p)_\Xi$ (resp. $G(\breve{\mathbb{Q}}_p)_\Xi$) denote the subgroup of $G(\mathbb{Q}_p)$ (resp. $G(\breve{\mathbb{Q}}_p)$) which fixes Ξ pointwise. By the main result of [5], there exists a smooth affine group scheme $\tilde{\mathcal{G}}_\Xi$ over $\mathbb{Z}_p$ with generic fiber G which is uniquely characterised by the property $\tilde{\mathcal{G}}_\Xi(\breve{\mathbb{Z}}_p) = G(\breve{\mathbb{Q}}_p)_\Xi$. We call such a group scheme the Bruhat–Tits stabiliser group scheme associated to Ξ. If $\Xi = \{x\}$ is a point, we write $G(\mathbb{Q}_p)_x$ (resp. $\tilde{\mathcal{G}}_x$) for $G(\mathbb{Q}_p)_{\{x\}}$ (resp. $\tilde{\mathcal{G}}_{\{x\}}$).

For $\Xi \subset B(G,\mathbb{Q}_p)$ as above, we let $\mathcal{G}_x$ denote the 'connected stabiliser' (cf. [5, §4]). We are mainly interested in the case that Ξ is a point or an open facet $\mathfrak{f}$. In this case $\mathcal{G}_\mathfrak{f}$ (resp. $\mathcal{G}_x$) is the parahoric group scheme associated to $\mathfrak{f}$ (resp. x).

We may also consider the corresponding objects over $\breve{\mathbb{Q}}_p$ and we use the same notation in this case. When it is understood which point of $B(G,\mathbb{Q}_p)$ or $B(G,\breve{\mathbb{Q}}_p)$ we are referring to, we simply write $\tilde{\mathcal{G}}$ and $\mathcal{G}$ for the corresponding group schemes.

An important case that we need for applications is when $\tilde{\mathcal{G}}_x = \mathcal{G}_x$, i.e. the parahoric is equal to the Bruhat–Tits stabiliser. When this happens, we necessarily have $\tilde{\mathcal{G}}_\mathfrak{f} = \tilde{\mathcal{G}}_x$, where $\mathfrak{f}$ is the facet containing x, and $x \in \mathfrak{f}$ is a point 'in general position'. A parahoric group scheme $\mathcal{G}$ over $\mathbb{Z}_p$ (resp. $\breve{\mathbb{Z}}_p$) is called a *connected parahoric* if there exists $x \in B(G,\mathbb{Q}_p)$ (resp. $x \in B(G,\breve{\mathbb{Q}}_p)$) such that $\mathcal{G} = \mathcal{G}_x = \tilde{\mathcal{G}}_x$.

Let $\pi_1(G)$ be the algebraic fundamental group of $G_{\mathbb{Q}_p}$, equipped with its action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ (cf. the introduction of [3]), and let $I \subset \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ be the inertia group.

Lemma 2.2.2. *If $\pi(G)_I$ is torsion free, then $\tilde{\mathcal{G}}_x = \mathcal{G}_x$ for all x. In other words, all parahoric group schemes are connected parahoric group schemes.*

Proof. This follows from [41, Remark 11 of the appendix]. □

2.2.3. Let $S \subset G_{\breve{\mathbb{Q}}_p}$ be a maximal $\breve{\mathbb{Q}}_p$-split torus defined over $\mathbb{Q}_p$ and let T be its centraliser; it is a maximal torus of G because $G_{\breve{\mathbb{Q}}_p}$ is quasi-split by a theorem of Steinberg. Choose a σ-invariant alcove $\mathfrak{a}$ in the apartment of $B(G,\breve{\mathbb{Q}}_p)$ associated to S. Let N be the normaliser of T in $G_{\breve{\mathbb{Q}}_p}$. We define the *relative Weyl group* as

$$W_0 := N(\breve{\mathbb{Q}}_p)/T(\breve{\mathbb{Q}}_p)$$

and the *Iwahori–Weyl group* (or extended affine Weyl group) as

$$\tilde{W} := N(\breve{\mathbb{Q}}_p)/\mathcal{T}(\breve{\mathbb{Z}}_p),$$

where $\mathcal{T}$ over $\breve{\mathbb{Z}}_p$ is the connected Néron model of T. There is a short exact sequence

$$0 \to X_*(T)_I \to \tilde{W} \to W_0 \to 0,$$

where I is the inertia group and $X_*(T)_I$ denotes the inertia coinvariants of the cocharacter lattice $X_*(T)$ of T. The map $X_*(T)_I \to \tilde{W}$ is denoted on elements by $\lambda \mapsto t^\lambda$. Let $\mathbb{S} \subset \tilde{W}$ denote the set of simple reflections in the walls of $\mathfrak{a}$ and let $\tilde{W}_a$ be the subgroup of $\tilde{W}$ generated by $\mathbb{S}$, which we will call the *affine Weyl group*.

Parahoric subgroups $\mathcal{K}$ of $G(\breve{\mathbb{Q}}_p)$ that contain the Iwahori subgroup corresponding to $\mathfrak{a}$ are called *standard parahoric subgroups*; they correspond to subsets $K \subset \mathbb{S}$ such that the subgroup W_K generated by K is finite; we will call such subsets *types*. This identification is Frobenius equivariant in the sense that $\sigma(\mathcal{K})$ corresponds to $\sigma(K)$. In particular, a subset $K \subset \mathbb{S}$ corresponds to a parahoric subgroup of G if and only if $\sigma(K) = K$; note that our fixed Iwahori subgroup corresponds to $\emptyset \subset \mathbb{S}$. There

are parahoric group schemes $\mathcal{G}_K$ over $\breve{\mathbb{Z}}_p$ associated to types K as above, and we have identifications $\sigma^*\mathcal{G}_K \simeq \mathcal{G}_{\sigma(K)}$. In particular, if K is stable under σ, then $\mathcal{G}_K$ is defined over $\mathbb{Z}_p$. The maximal reductive quotient $(\overline{\mathcal{G}}_K)^{\mathrm{red}}$ of the special fiber $\overline{\mathcal{G}}_K$ of $\mathcal{G}_K$ is a split reductive group over the residue field k of $\breve{\mathbb{Q}}_p$, and the image of $\mathcal{G}_\emptyset$ in $(\overline{\mathcal{G}}_K)^{\mathrm{red}}$ is a Borel subgroup. The set of simple roots of $(\overline{\mathcal{G}}_K)^{\mathrm{red}}$ with respect to this Borel subgroup can be identified with K.

Lemma 2.2.4. *Let $J \subset K \subset \mathbb{S}$ and suppose that $\mathcal{G}_K$ is a connected parahoric, then $\mathcal{G}_J$ is a connected parahoric.*

Proof. Let $x_K, x_J \in B(G, \breve{\mathbb{Q}}_p)$ such that $\mathcal{G}_K = \mathcal{G}_{x_K}$ and $\mathcal{G}_J = \mathcal{G}_{x_j}$. We assume that x_J and x_K are in general position in their respective facets. Then we have $\mathcal{G}_{x_K} = \tilde{\mathcal{G}}_{x_K}$ since $\mathcal{G}$ is a connected parahoric, and we have $\tilde{\mathcal{G}}_{x_J} = \tilde{\mathcal{G}}_{\mathfrak{f}_J}$, where $\mathfrak{f}_J$ is the facet corresponding to J.

Since x_K lies in the closure of $\mathfrak{f}_J$, it follows that $\tilde{\mathcal{G}}_{x_J}(\breve{\mathbb{Z}}_p) \subset \tilde{\mathcal{G}}_{x_K}(\breve{\mathbb{Z}}_p) = \mathcal{G}_{x_K}(\breve{\mathbb{Z}}_p)$. But $\mathcal{G}_{x_K}(\breve{\mathbb{Z}}_p)$ is contained in the kernel of the Kottwitz map $\kappa : G(\breve{\mathbb{Q}}_p) \to \pi_1(G)_I$. Therefore, we have $\tilde{\mathcal{G}}_{x_J}(\breve{\mathbb{Z}}_p) \subset \ker(\kappa)$ and hence we deduce as in Lemma 2.2.2 that $\tilde{\mathcal{G}}_{x_J} = \mathcal{G}_{x_J}$. □

There is a split short exact sequence (our choice of $\mathfrak{a}$ provides a splitting)

$$0 \to \tilde{W}_a \to \tilde{W} \to \pi_1(G)_I \to 0. \tag{2.2.1}$$

The affine Weyl group $\tilde{W}_a$ has the structure of a Coxeter group, and we will use this to define a Bruhat order (denoted by $\leq$) and a notion of length on $\tilde{W}$, by splitting (2.2.1) and regarding $\pi_1(G)_I \subset \tilde{W}$ as the subset of length zero elements. We will write $\ell(w)$ for the length of an element of $\tilde{W}$. Similarly, we define a partial order $\leq$ on $W_K \backslash \tilde{W} / W_K$ by taking minimal length representatives of double cosets.

2.2.5. In this section we will recall some definitions from [50, 54, 57] and state some results. Let the notation be as in Sections 2.2.1 and 2.2.3, so in particular G denotes a connected reductive group over $\mathbb{Q}_p$. Let $\mathcal{G}_K$ be a parahoric group scheme over $\breve{\mathbb{Z}}_p$ corresponding to a σ-stable type $K \subset \mathbb{S}$. For an object R of $\mathbf{Aff}_k^{\mathrm{perf}}$ we set

$$D_R = \operatorname{Spec} W(R), \qquad D_R^* = \operatorname{Spec} W(R)[1/p],$$

where $W(R)$ denotes the ring of p-typical Witt vectors of R. We define group-valued functors on $\mathbf{Aff}_k^{\mathrm{perf}}$ sending an object R to

$$\begin{aligned} LG(R) &:= G(D_R^*) \\ L^+\mathcal{G}_K(R) &:= \mathcal{G}_K(D_R) \\ L^m\mathcal{G}_K(R) &:= \mathcal{G}_K\left(W(R)/p^m W(R)\right), \end{aligned}$$

which we call the *loop group*, respectively the *positive loop group*, respectively the *m-truncated loop group*. It follows from [57, Section 1.1] that $L^m\mathcal{G}_K$ and $L^+\mathcal{G}_K$ are representable by perfect schemes over k and that $L^+\mathcal{G}_K = \varprojlim_m L^m\mathcal{G}_K$. Moreover, [57, Proposition 1.1] tells us that LG is representable by an ind-perfect ind-scheme, which means that it is isomorphic to an inductive limit of perfect schemes along closed immersions. By [57, Lemma 1.2.(i)], the natural map $L^+\mathcal{G}_K \to LG$ is a closed immersion.

2.2.6. Let R be a perfect $\overline{\mathbb{F}}_p$-algebra and let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{G}_K$-torsors on D_R.[6] Recall from [54, Section 3.1.3] that a *modification* $\beta : \mathcal{E} \dashrightarrow \mathcal{F}$ is an isomorphism of G-torsors

$$\beta : \mathcal{E}\big|_{D_R^*} \to \mathcal{F}\big|_{D_R^*}.$$

[6]Here we mean torsor in the fpqc topology on $D_R = \operatorname{Spec} W(R)$ in the usual way.

It follows from the proof of [57, Lemma 1.3] that $\mathcal{E}$ can be trivialised after an étale cover $\operatorname{Spec} D_{R'} \to \operatorname{Spec} D_R$ coming from an étale cover $\operatorname{Spec} R' \to \operatorname{Spec} R$. Therefore we can also think of $\mathcal{G}_K$-torsors over D_R as étale $L^+\mathcal{G}_K$-torsors over $\operatorname{Spec} R$.

We define the (partial) *affine flag variety* Gr_K to be the functor sending R to the set of isomorphism classes of modifications

$$\alpha : \mathcal{E} \dashrightarrow \mathcal{E}^0,$$

where $\mathcal{E}$ is a $\mathcal{G}_K$-torsor over D_R, and where $\mathcal{E}^0$ is the trivial $\mathcal{G}_K$-torsor over D_R. There is a natural action of

$$LG = \mathrm{Aut}(\mathcal{E}^0|_{D_R^*})$$

on Gr_K by postcomposing α with an automorphism of the restriction to D_R^* of $\mathcal{E}^0$, and the orbit of the $\overline{\mathbb{F}}_p$-point of Gr_K given by the identity modification $\mathcal{E}^0 \to \mathcal{E}^0$ induces a map $O : LG \to \mathrm{Gr}_K$. The map O induces an identification (that we will implicitly use from now on)

$$\mathrm{Gr}_K(\overline{\mathbb{F}}_p) \simeq G(\breve{\mathbb{Q}}_p)/\mathcal{G}_K(\breve{\mathbb{Z}}_p).$$

It is a result of [2, 57] that Gr_K is representable by an ind-projective ind-scheme. We also define the *Hecke stack* Hk_K to be the stack on $\mathbf{Aff}_k^{\mathrm{perf}}$ sending R to the groupoid of modifications $\beta : \mathcal{E} \dashrightarrow \mathcal{F}$. The natural map $\mathrm{Gr}_K \to \mathrm{Hk}_K$ is an $L^+\mathcal{G}_K$-torsor for the étale topology, where $L^+\mathcal{G}_K$ acts on Gr_K via the closed immersion $L^+\mathcal{G}_K \subset LG$.

2.2.7. Recall from [50, Definition 4.1.3] that a (local) $\mathcal{G}_K$-*shtuka* over a perfect k-algebra R is a pair $(\mathcal{E}, \beta)$, where $\mathcal{E}$ is a $\mathcal{G}_K$-torsor over D_R and where β is a modification $\beta : \sigma^*\mathcal{E} \dashrightarrow \mathcal{E}$. Here $\sigma : D_R \to D_R$ denotes the Frobenius morphism induced from the absolute Frobenius on R, and we consider the restriction of $\sigma^*\mathcal{E}$ to D_R^* as a G-torsor via the isomorphism $\sigma : \sigma^*G \to G$, coming from the fact that G is defined over $\mathbb{Q}_p$. A morphism of shtukas $(\mathcal{E}, \beta) \to (\mathcal{E}', \beta')$ is an isomorphism $f : \mathcal{E} \to \mathcal{E}'$ of $\mathcal{G}_K$-torsors such that the following diagram commutes

$$\begin{array}{ccc} \sigma^*\mathcal{E} & \overset{\beta}{\dashrightarrow} & \mathcal{E} \\ \downarrow{\scriptstyle \sigma^* f} & & \downarrow{\scriptstyle f} \\ \sigma^*\mathcal{E}' & \overset{\beta'}{\dashrightarrow} & \mathcal{E}'. \end{array}$$

We will write $\mathrm{Sht}_{G,K}(R)$ for the groupoid of $\mathcal{G}_K$-shtukas over R and $\mathrm{Sht}_{G,K}$ for the stack on $\mathbf{Aff}_k^{\mathrm{perf}}$ sending R to $\mathrm{Sht}_{G,K}(R)$.

Remark 2.2.8. When $\mathcal{G}_K = \mathrm{GL}_{n,\mathbb{Z}_p}$, a $\mathcal{G}_K$-shtuka over a perfect ring R is a projective module M of rank n over $W(R)$ together with an isomorphism

$$\beta : \sigma^*M[1/p] \to M[1/p].$$

If the map β satisfies $pM \subset \beta(\sigma^*M) \subset M$, then the pair (M, β) is a Dieudonné module. By a result of Gabber, see [34], there is a p-divisible group over $\operatorname{Spec} R$ with Dieudonné module (M, β).

2.2.9. For an inclusion of types $J \subset K$, there is a closed immersion $L^+\mathcal{G}_J \subset L^+\mathcal{G}_K$, since both are closed subschemes of LG by [57, Lemma 1.2.(i)]. If J and K are σ-stable, then pushing out torsors along $L^+\mathcal{G}_J \to L^+\mathcal{G}_K$ induces a forgetful map

$$\mathrm{Sht}_{G,J} \to \mathrm{Sht}_{G,K}\,.$$

In this section we will show that these forgetful maps are representable by perfectly proper algebraic spaces, which is an analogue of [41, Proposition 8.7].

Let $(\overline{\mathcal{G}}_K)^{\mathrm{red}}$ be the maximal reductive quotient of the 1-truncated loop group $L^1\mathcal{G}_K = \overline{\mathcal{G}}_K$ and let H_J be the image of $\mathcal{G}_J$ in $(\overline{\mathcal{G}}_K)^{\mathrm{red}}$; it is a standard parabolic subgroup of type $J \subset K$ (recall that K can be identified with the set of simple roots of $(\overline{\mathcal{G}}_K)^{\mathrm{red}}$ with respect to the Borel B that is the image of $\mathcal{G}_\emptyset \to (\overline{\mathcal{G}}_K)^{\mathrm{red}}$). Recall that for a perfect group scheme H we write $\mathbf{B}H$ for the classifying stack of H; in other words, $\mathbf{B}H$ is the groupoid valued functor that sends an object R of $\mathbf{Aff}_k^{\mathrm{perf}}$ to the groupoid of H torsors (in the étale topology) over $\operatorname{Spec} R$. There is a natural morphism $\operatorname{Spec} k \to \mathbf{B}H$ corresponding to the trivial H-torsor over $\operatorname{Spec} k$, which induces an isomorphism $[\operatorname{Spec} k/H] \to \mathbf{B}H$.

Lemma 2.2.10. *The forgetful map $\mathbf{B}L^+\mathcal{G}_J \to \mathbf{B}L^+\mathcal{G}_K$ is a $(\overline{\mathcal{G}}_K)^{\mathrm{red}}/H_J$-bundle for the étale topology, in particular it is representable by perfectly proper algebraic spaces.*

Proof. Let R be a perfect $\overline{\mathbb{F}}_p$-algebra and let X be an $L^+\mathcal{G}_K$ torsor over $\operatorname{Spec} R$ represented by a map $\operatorname{Spec} R \to \mathbf{B}L^+\mathcal{G}_K$. It follows from the definition of quotient stacks that both squares in the following diagram of stacks are Cartesian

$$\begin{array}{ccc} X & \longrightarrow & \operatorname{Spec} k \\ \downarrow & & \downarrow \\ [X/L^+\mathcal{G}_J] & \longrightarrow & \mathbf{B}L^+\mathcal{G}_J \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \longrightarrow & \mathbf{B}L^+\mathcal{G}_K. \end{array}$$

By [57, Lemma 1.3], there is an étale cover $T \to \operatorname{Spec} R$ such that X_T is isomorphic to the trivial $L^+\mathcal{G}_K$ torsor over T, hence $[X/L^+\mathcal{G}_J]$ is étale locally isomorphic to $\operatorname{Spec} R \times [L^+\mathcal{G}_K/L^+\mathcal{G}_J]$. Therefore it suffices to show that $[L^+\mathcal{G}_K/L^+\mathcal{G}_J]$ is representable by a perfectly proper scheme. It follows from the proof of [41, Proposition 8.7], cf. Lemma 2.3.5 below, that

$$[L^+\mathcal{G}_K/L^+\mathcal{G}_J] \simeq [(\overline{\mathcal{G}}_K)^{\mathrm{red}}/H_J]$$

and the latter is representable by a perfectly proper scheme because it is the perfection of a partial flag variety for $(\overline{\mathcal{G}}_K)^{\mathrm{red}}$. □

Corollary 2.2.11. *The map $\mathrm{Sht}_{G,J} \to \mathrm{Sht}_{G,K}$ is a $(\overline{\mathcal{G}}_K)^{\mathrm{red}}/H_J$-bundle for the étale topology, in particular it is representable by perfectly proper algebraic spaces.*

Proof. This follows because the following diagram is Cartesian

$$\begin{array}{ccc} \mathrm{Sht}_{G,J} & \longrightarrow & \mathrm{Sht}_{G,K} \\ \downarrow & & \downarrow \\ \mathbf{B}L^+\mathcal{G}_J & \longrightarrow & \mathbf{B}L^+\mathcal{G}_K. \end{array} \tag{2.2.2}$$

Indeed, this is a straightforward consequence of the definitions. [A $\mathcal{G}_J$-shtuka is the same thing as a $\mathcal{G}_K$-shtuka $(\mathcal{E}, \beta)$ together with an $L^+\mathcal{G}_J$-torsor $\mathcal{E}'$ and an isomorphism $\alpha : \mathcal{E}' \times_{L^+\mathcal{G}_J} L^+\mathcal{G}_K \simeq \mathcal{E}$.] □

2.2.12. *Relative position.* It follows from the discussion in [19, Section 3.6] that there is an $L^+\mathcal{G}_K$-equivariant stratification

$$\mathrm{Gr}_K = \bigcup_{w \in W_K \backslash \tilde{W} / W_K} \mathrm{Gr}_K(w),$$

where each $\mathrm{Gr}_K(w)$ is a locally closed subscheme of Gr_K, such that on $\overline{\mathbb{F}}_p$-points we recover the Bruhat–Tits decomposition

$$\mathrm{Gr}_K(\overline{\mathbb{F}}_p) = G(\breve{\mathbb{Q}}_p)/\mathcal{G}_K(\breve{\mathbb{Z}}_p) = \bigcup_{w\in W_K\backslash\tilde{W}/W_K} \mathcal{G}_K(\breve{\mathbb{Z}}_p)\dot{w}\mathcal{G}_K(\breve{\mathbb{Z}}_p)/\mathcal{G}_K(\breve{\mathbb{Z}}_p),$$

see [41, Proposition 8 of the appendix]. We deduce from this that we get a decomposition

$$\left[L^+\mathcal{G}_K\backslash \mathrm{Gr}_K\right] =: \mathrm{Hk}_K = \bigcup_{w\in W_K\backslash\tilde{W}/W_K} \mathrm{Hk}_K(w),$$

where $\mathrm{Hk}_K(w) = [L^+\mathcal{G}_K\backslash \mathrm{Gr}_K(w)]$. It follows moreover from [19, Section 3.6] that the closure of $\mathrm{Gr}_K(w)$ is equal to

$$\mathrm{Gr}_K(\leq w) := \bigcup_{w'\leq w} \mathrm{Gr}_K(w').$$

Furthermore, when $K = \emptyset$, the dimension of $\mathrm{Gr}_K(w)$ is equal to the length $\ell(w)$ of w.

2.2.13. Let $\{\mu\}$ be a $G(\overline{\mathbb{Q}}_p)$-conjugacy class of cocharacters of $G_{\overline{\mathbb{Q}}_p}$. Recall that we fixed a maximal torus T of G in Section 2.2.3. Choose a Borel B of $G_{\breve{\mathbb{Q}}_p}$ containing $T_{\breve{\mathbb{Q}}_p}$ and let $\overline{\mu}$ be the image in $X_*(T)_I$ of a B-dominant representative of $\{\mu\}$. The set of $\{\mu\}$-admissible elements is defined as

$$\mathrm{Adm}(\{\mu\}) = \{w \in \tilde{W} \ : \ w \leq t^{x(\overline{\mu})} \text{ for some } x \in W_0\}.$$

There is a unique element $\tau = \tau_\mu \in \mathrm{Adm}(\{\mu\})$ of length zero and in fact $\mathrm{Adm}(\{\mu\}) \subset \tilde{W}_a\tau$. For K a σ-stable type, we define $\mathrm{Adm}(\{\mu\})_K$ as the image of $\mathrm{Adm}(\{\mu\})$ under $\tilde{W} \to W_K\backslash\tilde{W}/W_K$. We write ${}^K\mathrm{Adm}(\{\mu\})$ for $\mathrm{Adm}(\{\mu\}) \cap {}^K\tilde{W}$, where ${}^K\tilde{W} \subset \tilde{W}$ denotes the subset of elements that are of minimal length in their left W_K-coset.

If $\{\mu\}$ is minuscule and K is a σ-stable type, then we define the *perfect local model* attached to $\{\mu\}$ and K to be the perfect projective scheme

$$\mathrm{M}^{\mathrm{loc}}_{K,\{\mu\}} := \bigcup_{w\in\mathrm{Adm}(\{\mu\})_K} \mathrm{Gr}_K(w).$$

The scheme $\mathrm{M}^{\mathrm{loc}}_{K,\{\mu\}}$ is equidimensional of dimension $d = \langle 2\rho, \overline{\mu}\rangle = \ell(t^{x(\overline{\mu})})$, where ρ is the half sum of the positive roots (with respect to B). The locally closed subschemes $\mathrm{Gr}_K(w) \subset \mathrm{M}^{\mathrm{loc}}_{K,\{\mu\}}$ are called *Kottwitz–Rapoport (KR) strata*.

2.2.14. Let $\{\mu\}$ be a conjugacy class of cocharacters of $G_{\overline{\mathbb{Q}}_p}$ as above, and let $\mathrm{Adm}(\{\mu\})_K$ be the μ-admissible set. Recall that the stack Hk_K is the moduli stack of modifications $\mathcal{E} \dashrightarrow \mathcal{F}$ of $L^+\mathcal{G}_K$-torsors. We define a map $\mathrm{Rel} : \mathrm{Sht}_{G,K} \to \mathrm{Hk}_K$ sending $(\mathcal{E}, \beta)$ to $\beta : \sigma^*\mathcal{E} \dashrightarrow \mathcal{E}$. For $w \in W_K\backslash\tilde{W}/W_K$, we have the locally closed substack $\mathrm{Hk}_K(w) \subset \mathrm{Hk}_K$ from Section 2.2.12, and its pullback along Rel defines a locally closed substack

$$\mathrm{Sht}_{G,K}(w) \subset \mathrm{Sht}_{G,K}\,.$$

Following [50, Definition 4.1.3], we define the stack of shtukas of level $\mathcal{G}_K$ and type μ to be

$$\mathrm{Sht}_{G,K,\{\mu\}} := \bigcup_{w\in\mathrm{Adm}(\{\mu\})_K} \mathrm{Sht}_{G,K}(w);$$

it is a closed substack of $\mathrm{Sht}_{G,K}$ by the discussion in Section 2.2.12. If $J \subset K$ is another σ-stable type, then the following diagram commutes by definition of $\mathrm{Adm}(\{\mu\})_J$ and $\mathrm{Adm}(\{\mu\})_K$ (but it is generally not Cartesian)

$$\begin{array}{ccc} \mathrm{Sht}_{G,J,\{\mu\}} & \hookrightarrow & \mathrm{Sht}_{G,J} \\ \downarrow & & \downarrow \\ \mathrm{Sht}_{G,K,\{\mu\}} & \hookrightarrow & \mathrm{Sht}_{G,K}\,. \end{array}$$

Proposition 2.2.15. The forgetful morphism $\mathrm{Sht}_{G,J,\{\mu\}} \to \mathrm{Sht}_{G,K,\{\mu\}}$ is representable by perfectly proper algebraic spaces.

Proof. We know that $\mathrm{Sht}_{G,J,\{\mu\}} \to \mathrm{Sht}_{G,J}$ is representable by perfectly proper algebraic spaces because it is a closed immersion, and the map $\mathrm{Sht}_{G,J} \to \mathrm{Sht}_{G,K}$ is representable by perfectly proper algebraic spaces by Corollary 2.2.11. The composition is thus representable by perfectly proper algebraic spaces and factors over $\mathrm{Sht}_{G,K,\{\mu\}}$, which proves the result. □

2.3. **Restricted local shtukas and forgetful maps.** We will recall some results from [50, Section 4.2]. Fix a geometric conjugacy class of minuscule cocharacters $\{\mu\}$ of $G_{\overline{\mathbb{Q}}_p}$ for the rest of this section, and let $\mathrm{Adm}(\{\mu\})_K$ be the μ-admissible set. Recall from [50, Lemma 4.1.4] that $\mathrm{Sht}_{G,K}$ has the following quotient description: Let $\sigma : L^+\mathcal{G}_K \to L^+\mathcal{G}_K$ be the relative Frobenius morphism and let $L^+\mathcal{G}_K$ act on LG via $h \cdot g = (h^{-1}g\sigma(h))$, we denote this action by Ad_σ. With this notation, there is an isomorphism

$$\mathrm{Sht}_{G,K} \simeq \left[\frac{LG}{\mathrm{Ad}_\sigma\, L^+\mathcal{G}_K}\right].$$

The map $\Lambda : LG \to \mathrm{Sht}_{G,K}$ constructed this way corresponds to a shtuka over LG: It is the modification $\beta : \mathcal{E}^0_{LG} \simeq \sigma^*\mathcal{E}^0_{LG} \dashrightarrow \mathcal{E}^0_{LG}$ given by the tautological element in LG. Moreover the map $LG \to \mathrm{Sht}_{G,K}$ is precisely the universal $L^+\mathcal{G}_K$-torsor over $\mathrm{Sht}_{G,K}$.

Consider the following commutative diagram

$$\begin{array}{ccc} LG & \xrightarrow{\;O\;} & \mathrm{Gr}_K \\ \downarrow{\scriptstyle \Lambda} & & \downarrow \\ \mathrm{Sht}_{G,K} & \xrightarrow{\mathrm{Rel}} & \mathrm{Hk}_K\,. \end{array}$$

There is a closed subscheme $\mathrm{M}^{\mathrm{loc},\infty}_{K,\{\mu\}} \subset LG$ defined to be the inverse image of $\mathrm{M}^{\mathrm{loc}}_{K,\{\mu\}} \subset \mathrm{Gr}_K$ under $LG \to \mathrm{Gr}_K$. The discussion in the previous paragraph, along with the commutative diagram, tells us that there is a natural identification

$$\mathrm{Sht}_{G,K,\{\mu\}} \simeq \left[\frac{\mathrm{M}^{\mathrm{loc},\infty}_{K,\{\mu\}}}{\mathrm{Ad}_\sigma\, L^+\mathcal{G}_K}\right].$$

For $J \subset K$ a σ-stable subset, there is a closed immersion $\mathrm{M}^{\mathrm{loc},\infty}_{J,\{\mu\}} \subset \mathrm{M}^{\mathrm{loc},\infty}_{K,\{\mu\}}$ which identifies

$$\frac{\mathrm{M}^{\mathrm{loc},\infty}_{K,\{\mu\}}}{L^+\mathcal{G}_J} \subset \mathrm{Gr}_J$$

with the preimage of $\mathrm{M}^{\mathrm{loc}}_{K,\{\mu\}}$ under $\mathrm{Gr}_J \to \mathrm{Gr}_K$.

2.3.1. Let $\beta_K : L^+\mathcal{G}_K \to (\overline{\mathcal{G}}_K)^{\mathrm{red}}$ be the natural map, where $(\overline{\mathcal{G}}_K)^{\mathrm{red}}$ is the maximal reductive quotient of $\overline{\mathcal{G}}_K = L^1\mathcal{G}_K$. Define $\mathrm{M}^{\mathrm{loc},1-\mathrm{red}}_{K,\{\mu\}} := \ker\beta_K \backslash \mathrm{M}^{\mathrm{loc},\infty}_{K,\{\mu\}}$; it is a $(\overline{\mathcal{G}}_K)^{\mathrm{red}}$-torsor over $\mathrm{M}^{\mathrm{loc}}_{K,\{\mu\}}$. We then define

$$\mathrm{Sht}^{(\infty,1)}_{G,K,\{\mu\}} := \left[\frac{\mathrm{M}^{\mathrm{loc},1-\mathrm{red}}_{K,\{\mu\}}}{\mathrm{Ad}_\sigma\, L^+\mathcal{G}_K}\right].$$

It follows from Lemma 2.1.1 that the twisted conjugation action of $L^+\mathcal{G}_K$ on $\mathrm{M}^{\mathrm{loc},1-\mathrm{red}}_{K,\{\mu\}}$ factors through the action of $L^m\mathcal{G}_K$ for $m \gg 0$, and for such m we define the stack of (m)-restricted shtukas of type $\{\mu\}$ by

$$\mathrm{Sht}^{(m,1)}_{G,K,\{\mu\}} := \left[\frac{\mathrm{M}^{\mathrm{loc},1-\mathrm{red}}_{K,\{\mu\}}}{\mathrm{Ad}_\sigma\, L^m\mathcal{G}_K}\right].$$

Note that there are natural morphisms

$$\mathrm{Sht}_{G,K,\{\mu\}} = \left[\frac{\mathrm{M}^{\mathrm{loc},\infty}_{K,\{\mu\}}}{\mathrm{Ad}_\sigma\, L^+\mathcal{G}_K}\right] \to \left[\frac{\mathrm{M}^{\mathrm{loc},1-\mathrm{red}}_{K,\{\mu\}}}{\mathrm{Ad}_\sigma\, L^+\mathcal{G}_K}\right] \to \left[\frac{\mathrm{M}^{\mathrm{loc},1-\mathrm{red}}_{K,\{\mu\}}}{\mathrm{Ad}_\sigma\, L^m\mathcal{G}_K}\right] = \mathrm{Sht}^{(m,1)}_{G,K,\{\mu\}} \tag{2.3.1}$$

induced by the natural map $\mathrm{M}^{\mathrm{loc},\infty}_{K,\{\mu\}} \to \mathrm{M}^{\mathrm{loc},1-\mathrm{red}}_{K,\{\mu\}}$ and the natural map $L^+\mathcal{G}_K \to L^m\mathcal{G}_K$.

Remark 2.3.2. There is a 'local model diagram'

$$\begin{array}{ccc} & \mathrm{M}^{\mathrm{loc},1-\mathrm{red}}_{K,\{\mu\}} & \\ \swarrow & & \searrow \\ \mathrm{Sht}^{(m,1)}_{G,K,\{\mu\}} & & \mathrm{M}^{\mathrm{loc}}_{K,\{\mu\}}\,. \end{array}$$

The left-hand map is an $L^m\mathcal{G}_K$-torsor while the right-hand map is a $(\overline{\mathcal{G}}_K)^{\mathrm{red}}$-torsor. In particular, the stack $\mathrm{Sht}^{(m,1)}_{G,K,\{\mu\}}$ is an equidimensional pfp algebraic stack.[7] Indeed $\mathrm{M}^{\mathrm{loc}}_{K,\{\mu\}}$ is pfp and equidimensional and since the right-hand map is perfectly smooth of relative dimension $\dim(\overline{\mathcal{G}}_K)^{\mathrm{red}}$ we find that $\mathrm{M}^{\mathrm{loc},1-\mathrm{red}}_{K,\{\mu\}}$ is pfp and equidimensional by Lemma 2.1.6.

2.3.3. The goal of this section is to compare $\mathrm{Sht}^{(m,1)}_{G,K,\{\mu\}}$ and $\mathrm{Sht}^{(m',1)}_{G,\emptyset,\{\mu\}}$. Unfortunately, there is no natural map between them when $K \neq \emptyset$. However, we will be able to construct a correspondence between them instead, and study its properties, see Proposition 2.3.4, Lemma 2.3.9 and Section 2.3.10.

Consider the closed immersion $L^+\mathcal{G}_\emptyset \subset L^+\mathcal{G}_K$, which induces a closed immersion $B \subset (\overline{\mathcal{G}}_K)^{\mathrm{red}}$, where B is the image of $L^1\mathcal{G}_\emptyset$ in $(\overline{\mathcal{G}}_K)^{\mathrm{red}}$; let $\gamma : L^+\mathcal{G}_\emptyset \to B$ be the natural surjection. By Lemma 2.1.1, we can choose $m \gg 0$ such that the action $\mathrm{Ad}_\sigma\, L^+\mathcal{G}_K$ on $\ker\gamma\backslash \mathrm{M}^{\mathrm{loc},\infty}_{K,\{\mu\}}$ factors through $L^m\mathcal{G}_K$. As in equation (2.3.1), the natural maps

$$\begin{aligned} \mathrm{M}^{\mathrm{loc},\infty}_{K,\{\mu\}} &\to \ker\gamma\backslash \mathrm{M}^{\mathrm{loc},\infty}_{K,\{\mu\}} \\ L^+\mathcal{G}_K &\to L^m\mathcal{G}_K \end{aligned}$$

[7] A quotient stack $[X/G]$ is defined to be equidimensional if X is equidimensional. The dimension of $[X/G]$ is defined to be $\dim X - \mathrm{DimG}$ whenever both X and G are finite dimensional. This is well-defined in view of Lemma 2.1.16.

induce a natural map

$$\mathrm{Sht}_{G,K,\{\mu\}} = \left[\frac{\mathrm{M}^{\mathrm{loc},\infty}_{K,\{\mu\}}}{\mathrm{Ad}_\sigma\, L^+\mathcal{G}_K}\right] \to \left[\frac{\ker\gamma\backslash\, \mathrm{M}^{\mathrm{loc},\infty}_{K,\{\mu\}}}{\mathrm{Ad}_\sigma\, L^m\mathcal{G}_K}\right].$$

For m as above, let H_m be the image of $L^+\mathcal{G}_\emptyset$ in $L^m\mathcal{G}_K$. Since $\mathrm{M}^{\mathrm{loc},\infty}_{\emptyset,\{\mu\}} \subset \mathrm{M}^{\mathrm{loc},\infty}_{K,\{\mu\}}$, it follows that the action of $L^+\mathcal{G}_\emptyset \subset L^+\mathcal{G}_K$ on $\ker\gamma\backslash\, \mathrm{M}^{\mathrm{loc},\infty}_{\emptyset,\{\mu\}}$ factors through H_m. Therefore there is a natural map

$$\mathrm{Sht}_{G,\emptyset,\{\mu\}} = \left[\frac{\mathrm{M}^{\mathrm{loc},\infty}_{\emptyset,\{\mu\}}}{\mathrm{Ad}_\sigma\, L^+\mathcal{G}_\emptyset}\right] \to \left[\frac{\ker\gamma\backslash\, \mathrm{M}^{\mathrm{loc},\infty}_{\emptyset,\{\mu\}}}{\mathrm{Ad}_\sigma\, H_m}\right]$$

induced by $\mathrm{M}^{\mathrm{loc},\infty}_{\emptyset,\{\mu\}} \to \ker\gamma\backslash\, \mathrm{M}^{\mathrm{loc},\infty}_{\emptyset,\{\mu\}}$ and $L^+\mathcal{G}_\emptyset \to H_m$.

Proposition 2.3.4. *If m is an integer such that the action $\mathrm{Ad}_\sigma\, L^+\mathcal{G}_K$ on $\ker\gamma\backslash\, \mathrm{M}^{\mathrm{loc},\infty}_{K,\{\mu\}}$ factors through $L^m\mathcal{G}_K$, then the diagram*

$$\begin{array}{ccc} \mathrm{Sht}_{G,\emptyset,\{\mu\}} & \longrightarrow & \left[\frac{\ker\gamma\backslash\, \mathrm{M}^{\mathrm{loc},\infty}_{\emptyset,\{\mu\}}}{\mathrm{Ad}_\sigma\, H_m}\right] \\ \downarrow & & \downarrow \\ \mathrm{Sht}_{G,K,\{\mu\}} & \longrightarrow & \left[\frac{\ker\gamma\backslash\, \mathrm{M}^{\mathrm{loc},\infty}_{K,\{\mu\}}}{\mathrm{Ad}_\sigma\, L^m\mathcal{G}_K}\right], \end{array}$$

where the right vertical map is induced by the closed immersions $\mathrm{M}^{\mathrm{loc},\infty}_{\emptyset,\{\mu\}} \hookrightarrow \mathrm{M}^{\mathrm{loc},\infty}_{K,\{\mu\}}$ and $H_m \hookrightarrow L^m\mathcal{G}_K$, is Cartesian.

We start by proving a lemma.

Lemma 2.3.5. *Both squares in the following diagram of perfect group schemes are Cartesian.*

$$\begin{array}{ccccc} L^+\mathcal{G}_\emptyset & \twoheadrightarrow & H_m & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ L^+\mathcal{G}_K & \longrightarrow & L^m\mathcal{G}_K & \longrightarrow & (\overline{\mathcal{G}}_K)^{\mathrm{red}}. \end{array} \tag{2.3.2}$$

Proof. We first check that the outer square is Cartesian: It is enough to check this on k'-points for all algebraically closed fields k' because $L^+\mathcal{G}_\emptyset \to L^+\mathcal{G}_K$ is a closed immersion by [57, Lemma 1.2.(i)]. The result on the level of k'-points is [5, Theorem 4.6.33].

The left square is Cartesian by definition of H_m and it therefore follows from general properties of Cartesian squares that the right square is also Cartesian. □

Lemma 2.3.6. *The stacks*

$$\left[\frac{\ker\gamma\backslash\, \mathrm{M}^{\mathrm{loc},\infty}_{\emptyset,\{\mu\}}}{\mathrm{Ad}_\sigma\, H_m}\right] \quad \textit{and} \quad \left[\frac{\ker\gamma\backslash\, \mathrm{M}^{\mathrm{loc},\infty}_{K,\{\mu\}}}{\mathrm{Ad}_\sigma\, L^m\mathcal{G}_K}\right]$$

are equidimensional of the same dimension.

Proof. To compute the dimensions we note that it follows from the right Cartesian square in Lemma 2.3.5 that

$$\mathrm{Dim}\, H_m = \mathrm{Dim}\, L^m\mathcal{G}_K - (\mathrm{Dim}(\overline{\mathcal{G}}_K)^{\mathrm{red}} - \mathrm{Dim}\, B)$$

and thus it suffices to show that

$$\mathrm{Dim}\left(\ker\gamma\backslash \mathrm{M}^{\mathrm{loc},\infty}_{\emptyset,\{\mu\}}\right) = \mathrm{Dim}\left(\ker\gamma\backslash \mathrm{M}^{\mathrm{loc},\infty}_{K,\{\mu\}}\right) - \mathrm{Dim}(\overline{\mathcal{G}}_K)^{\mathrm{red}} + \mathrm{Dim}\, B.$$

The map $\ker\gamma\backslash \mathrm{M}^{\mathrm{loc},\infty}_{\emptyset,\{\mu\}} \to \mathrm{M}^{\mathrm{loc}}_{\emptyset,\{\mu\}}$ is a B-torsor by construction and $\ker\gamma\backslash \mathrm{M}^{\mathrm{loc},\infty}_{K,\{\mu\}} \to \mathrm{M}^{\mathrm{loc}}_{K,\{\mu\}}$ is a $(\overline{\mathcal{G}}_K)^{\mathrm{red}}$-torsor by construction, see (2.3.2). Therefore the equality above is equivalent to the equality

$$\mathrm{Dim}\,\mathrm{M}^{\mathrm{loc}}_{K,\{\mu\}} = \mathrm{Dim}\,\mathrm{M}^{\mathrm{loc}}_{\emptyset,\{\mu\}},$$

which is true, see Section 2.2.13. □

Proof of Proposition 2.3.4. Consider the following diagram, where the maps are defined as in (2.3.1)

$$\begin{array}{ccccccc}
\mathrm{Sht}_{G,\emptyset,\{\mu\}} & = & \left[\frac{\mathrm{M}^{\mathrm{loc},\infty}_{\emptyset,\{\mu\}}}{\mathrm{Ad}_\sigma\, L^+\mathcal{G}_\emptyset}\right] & \longrightarrow & \left[\frac{\ker\gamma\backslash \mathrm{M}^{\mathrm{loc},\infty}_{\emptyset,\{\mu\}}}{\mathrm{Ad}_\sigma\, L^+\mathcal{G}_\emptyset}\right] & \longrightarrow & \left[\frac{\ker\gamma\backslash \mathrm{M}^{\mathrm{loc},\infty}_{\emptyset,\{\mu\}}}{\mathrm{Ad}_\sigma\, H_m}\right] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Sht}_{G,K,\{\mu\}} & = & \left[\frac{\mathrm{M}^{\mathrm{loc},\infty}_{K,\{\mu\}}}{\mathrm{Ad}_\sigma\, L^+\mathcal{G}_K}\right] & \longrightarrow & \left[\frac{\ker\gamma\backslash \mathrm{M}^{\mathrm{loc},\infty}_{K,\{\mu\}}}{\mathrm{Ad}_\sigma\, L^+\mathcal{G}_K}\right] & \longrightarrow & \left[\frac{\ker\gamma\backslash \mathrm{M}^{\mathrm{loc},\infty}_{K,\{\mu\}}}{\mathrm{Ad}_\sigma\, L^m\mathcal{G}_K}\right].
\end{array}$$

It follows from Lemma 2.3.5 that $P^m := \mathrm{Ker}\,(L^+\mathcal{G}_K \to L^m\mathcal{G}_K)$ is contained in $L^+\mathcal{G}_\emptyset$ and that P^m is also equal to the kernel of $L^+\mathcal{G}_\emptyset \to H_m$. We deduce that the right square is Cartesian by general properties of quotient stacks (the right horizontal maps are P^m-gerbes in a compatible way). The middle square is Cartesian by general properties of quotient stacks (the middle horizontal maps are $\ker\gamma$-torsors in a compatible way). We deduce that the outer square of the diagram is Cartesian. □

2.3.7. In this section we record two more lemmas.

Lemma 2.3.8. *For each integer $m' \geq 1$ there is an integer $m \gg m'$ such that there is an inclusion $\ker\,(L^+\mathcal{G}_\emptyset \to H_m) \subset \ker\left(L^+\mathcal{G}_\emptyset \to L^{m'}\mathcal{G}_\emptyset\right)$ of closed subschemes of $L^+\mathcal{G}_\emptyset$.*

Proof. Fix m'. Recall that

$$L^+\mathcal{G}_K \simeq \varprojlim_m L^m\mathcal{G}_K$$
$$L^+\mathcal{G}_\emptyset \simeq \varprojlim_m L^m\mathcal{G}_\emptyset,$$

and the first of these equalities moreover implies that $L^+\mathcal{G}_\emptyset \simeq \varprojlim_m H_m$. The lemma now follows from Lemma 2.1.1. □

It follows from Lemma 2.3.8 that for each m' there is an $m \gg m'$ such that the natural map $L^+\mathcal{G}_\emptyset \to L^{m'}\mathcal{G}_\emptyset$ factors through the natural map $L^+\mathcal{G}_\emptyset \to H_m$ via a surjection $H_m \to L^{m'}\mathcal{G}_\emptyset$. Note moreover that $L^+\mathcal{G}_\emptyset \to L^1\mathcal{G}_\emptyset = \overline{\mathcal{G}}_\emptyset \to (\overline{\mathcal{G}}_\emptyset)^{\mathrm{red}}$ factors through $L^+\mathcal{G}_\emptyset \to L^1\mathcal{G}_\emptyset \to B$ because the maximal reductive quotient of $\overline{\mathcal{G}}_\emptyset$ is isomorphic to the maximal reductive quotient of B. Thus there is a natural map $\ker\gamma \to \ker(L^+\mathcal{G}_\emptyset \to (\overline{\mathcal{G}}_\emptyset)^{\mathrm{red}})$ which induces a map $\ker\gamma\backslash \mathrm{M}^{\mathrm{loc},\infty}_{\emptyset,\{\mu\}} \to \mathrm{M}^{\mathrm{loc},1-\mathrm{red}}_{\emptyset,\{\mu\}}$. Recall moreover that for $m \gg 0$ the action $\mathrm{Ad}_\sigma\, L^+\mathcal{G}_\emptyset$ on $\mathrm{M}^{\mathrm{loc},1-\mathrm{red}}_{\emptyset,\{\mu\}}$ factors through an action of $L^{m'}\mathcal{G}_\emptyset$.

Lemma 2.3.9. *Let $m' \gg 0$ be a positive integer and let $m \gg m'$ satisfy the conclusion of Lemma 2.3.8. Then the map (induced by $\ker\gamma\backslash \mathrm{M}^{\mathrm{loc},\infty}_{\emptyset,\{\mu\}} \to \mathrm{M}^{\mathrm{loc},1-\mathrm{red}}_{\emptyset,\{\mu\}}$ and $H_m \to L^{m'}\mathcal{G}_\emptyset$)*

$$\left[\frac{\ker\gamma\backslash \mathrm{M}^{\mathrm{loc},\infty}_{\emptyset,\{\mu\}}}{\mathrm{Ad}_\sigma\, H_m}\right] \to \left[\frac{\mathrm{M}^{\mathrm{loc},1-\mathrm{red}}_{\emptyset,\{\mu\}}}{\mathrm{Ad}_\sigma\, L^{m'}\mathcal{G}_\emptyset}\right] = \mathrm{Sht}^{(m',1)}_{G,\emptyset,\{\mu\}}$$

is weakly perfectly smooth.

Proof. The natural map

$$\left[\frac{\ker\gamma\backslash \mathrm{M}^{\mathrm{loc},\infty}_{\emptyset,\{\mu\}}}{\mathrm{Ad}_\sigma H_m}\right] \to \left[\frac{\mathrm{M}^{\mathrm{loc},1-\mathrm{red}}_{\emptyset,\{\mu\}}}{\mathrm{Ad}_\sigma H_m}\right] = \mathrm{Sht}^{(m',1)}_{G,\emptyset,\{\mu\}}$$

is a torsor for $\ker(B \to (\overline{\mathcal{G}}_\emptyset)^{\mathrm{red}})$ and hence weakly perfectly smooth. The natural map

$$\left[\frac{\mathrm{M}^{\mathrm{loc},1-\mathrm{red}}_{\emptyset,\{\mu\}}}{\mathrm{Ad}_\sigma H_m}\right] \to \left[\frac{\mathrm{M}^{\mathrm{loc},1-\mathrm{red}}_{\emptyset,\{\mu\}}}{\mathrm{Ad}_\sigma L^{m'}\mathcal{G}_\emptyset}\right]$$

is a gerbe for $\ker(H_m \to L^{m'}\mathcal{G}_\emptyset)$ and is thus weakly perfectly smooth. It follows that the composition is weakly perfectly smooth, and the lemma is proved. □

2.3.10. The stack $\left[\frac{\ker\gamma\backslash \mathrm{M}^{\mathrm{loc},\infty}_{K,\{\mu\}}}{\mathrm{Ad}_\sigma L^m\mathcal{G}_K}\right]$ is not a stack of restricted shtukas in the sense of Shen–Yu–Zhang [50]. However, it is closely related to the more general stacks of restricted shtukas introduced in [54, Section 5.3]. We define for $n \geq 2$ the quotient

$$\mathrm{M}^{\mathrm{loc,n}}_{K,\{\mu\}} := \ker\left(L^+\mathcal{G}_K \to L^n\mathcal{G}_K\right)\backslash \mathrm{M}^{\mathrm{loc},\infty}_{K,\{\mu\}}.$$

Then by Lemma 2.1.1, for $m \gg n$ the action $\mathrm{Ad}_\sigma L^+\mathcal{G}_K$ on $\mathrm{M}^{\mathrm{loc,n}}_{K,\{\mu\}}$ will factor through $L^m\mathcal{G}_K$ and we define

$$\mathrm{Sht}^{(m,n),\mathrm{loc}}_{G,K,\{\mu\}} := \left[\frac{\mathrm{M}^{\mathrm{loc,n}}_{K,\{\mu\}}}{\mathrm{Ad}_\sigma L^m\mathcal{G}_K}\right].$$

We have added the 'loc' in the superscript and the condition that $n \geq 2$ so that these are not confused with the previously introduced stacks of restricted shtukas (since the notation is *not* compatible).

The proof of Lemma 2.3.8 shows that for $n \gg 0$ we have an inclusion $\ker\left(L^+\mathcal{G}_K \to L^n\mathcal{G}_K\right) \subset \ker\gamma$ and thus a natural map

$$\mathrm{M}^{\mathrm{loc,n}}_{K,\{\mu\}} \to \ker\gamma\backslash \mathrm{M}^{\mathrm{loc},\infty}_{K,\{\mu\}}.$$

This induces a morphism (for $m \gg n$ as before)

$$\mathrm{Sht}^{(m,n),\mathrm{loc}}_{G,K,\{\mu\}} = \left[\frac{\mathrm{M}^{\mathrm{loc,n}}_{K,\{\mu\}}}{\mathrm{Ad}_\sigma L^m\mathcal{G}_K}\right] \to \left[\frac{\ker\gamma\backslash \mathrm{M}^{\mathrm{loc},\infty}_{K,\{\mu\}}}{\mathrm{Ad}_\sigma L^m\mathcal{G}_K}\right],$$

which is a torsor for the image of $\ker\gamma$ in $L^n\mathcal{G}_K$, and thus perfectly smooth.

2.3.11. *The EKOR stratification.* Recall that ${}^K\mathrm{Adm}(\{\mu\})$ is the intersection of $\mathrm{Adm}(\{\mu\})$ with ${}^K\tilde{W}$, where ${}^K\tilde{W} \subset \tilde{W}$ denotes the subset of elements that are of minimal length in their left W_K-coset. By [50, Lemma 4.2.4], the underlying topological space of $\mathrm{Sht}^{(m,1)}_{G,K,\{\mu\}}$ is isomorphic to ${}^K\mathrm{Adm}(\{\mu\})$ equipped with the partial order topology (for the partial order $\preceq$ on ${}^K\mathrm{Adm}(\{\mu\})$ introduced in [50, page 3123]). They use this to define locally closed substacks $\mathrm{Sht}^{(m,1)}_{G,K,\{\mu\}}\{w\}$ for $w \in {}^K\mathrm{Adm}(\{\mu\})$ such that the locally closed substack

$$\mathrm{Sht}^{(m,1)}_{G,K,\{\mu\}}\{\preceq w\} := \bigcup_{w' \preceq w} \mathrm{Sht}^{(m,1)}_{G,K,\{\mu\}}\{w'\}.$$

is closed. This allows us to define the Ekedahl–Kottwitz–Oort–Rapoport (EKOR) stratification on any stack mapping to $\mathrm{Sht}^{(m,1)}_{G,K,\{\mu\}}$; for example on $\mathrm{Sht}_{G,K,\{\mu\}}$ via (2.3.1) and later on Shimura varieties of Hodge type. Note that if $K = \emptyset$ then the EKOR stratification agrees with the Kottwitz–Rapoport (KR) stratification from Section 2.2.13 and $\preceq$ agrees with $\leq$.

2.4. **Affine Deligne–Lusztig varieties.** Recall from [45, Section 2.3] that there is a partial order on the set $B(G)$ of σ-conjugacy classes in $G(\breve{\mathbb{Q}}_p)$. Let $\{\mu\}$ be a $G(\overline{\mathbb{Q}}_p)$-conjugacy class of cocharacters of $G_{\overline{\mathbb{Q}}_p}$ and let $B(G,\{\mu\}) \subset B(G)$ be the set of *neutral acceptable* σ-conjugacy classes with respect to $\{\mu\}$, see [48, Definition 2.5].

2.4.1. Let $\mathcal{E}$ be an LG-torsor over k', with k' an algebraically closed field of characteristic p, and let $\beta : \sigma^*\mathcal{E} \to \mathcal{E}$ be an isomorphism where σ is the absolute Frobenius. After choosing a trivialisation of $\mathcal{E}$, we see that β can be represented by an element $b_\beta \in G(W(k')[1/p])$ well-defined up to σ-conjugacy. Since the set of σ-conjugacy classes in $G(W(k')[1/p])$ does not depend on the choice of algebraically closed field k', it thus gives us an element $[b_\beta] \in B(G)$.

Let R be a perfect $\overline{\mathbb{F}}_p$-algebra, let $\mathcal{E}$ be an LG-torsor over R and let $\beta : \sigma^*\mathcal{E} \to \mathcal{E}$ be an isomorphism. Then for $[b] \in B(G)$, the subset (using the partial order introduced above)

$$(\operatorname{Spec} R)_{\leq [b]} := \{x \in \operatorname{Spec} R \,:\, [b_\beta(x)] \leq [b]\}$$

is closed in $\operatorname{Spec} R$ by [45, Theorem 3.6.(ii)] and

$$(\operatorname{Spec} R)_{[b]} := \{x \in \operatorname{Spec} R \,:\, [b_\beta(x)] = [b]\}$$

is locally closed. This gives us a stratification

$$\mathrm{Sht}_{G,K} := \bigcup_{[b]\in B(G)} \mathrm{Sht}_{G,K,[b]},$$

where $\mathrm{Sht}_{G,K,[b]}$ denotes the locally closed substack of $\mathrm{Sht}_{G,K}$ whose R-points are given by the full subgroupoid

$$\mathrm{Sht}_{G,K,[b]}(R) \subset \mathrm{Sht}_{G,K}(R)$$

of maps $\operatorname{Spec} R \to \mathrm{Sht}_{G,K}$ such that $(\operatorname{Spec} R)_{[b]} = \operatorname{Spec} R$. We will write $\mathrm{Sht}_{G,K,\{\mu\},[b]}$ for the intersection (fiber product over $\mathrm{Sht}_{G,K}$) of $\mathrm{Sht}_{G,K,\{\mu\}}$ and $\mathrm{Sht}_{G,K,[b]}$; we will see in Corollary 2.4.6 that this is non-empty if and only if $[b] \in B(G,\{\mu\})$.

2.4.2. Let $K \subset \mathbb{S}$ be a σ-stable type and let $b \in G(\breve{\mathbb{Q}}_p)$. Then we define the affine Deligne–Lusztig set

$$X(\mu,b)_K = \{g \in G(\breve{\mathbb{Q}}_p)/\mathcal{G}_K(\breve{\mathbb{Z}}_p) \mid g^{-1}b\sigma(g) \in \bigcup_{w\in \mathrm{Adm}(\{\mu\})_K} \mathcal{G}_K(\breve{\mathbb{Z}}_p)\dot{w}\mathcal{G}_K(\breve{\mathbb{Z}}_p)\}.$$

Let J_b be the algebraic group over $\mathbb{Q}_p$ whose R-points are given by

$$J_b(R) = \{g \in G(\breve{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} R) \mid g^{-1}b\sigma(g) = b\}.$$

Then $J_b(\mathbb{Q}_p) \subset G(\breve{\mathbb{Q}}_p)$ acts on $X(\mu,b)_K$ via left multiplication. By [18, Theorem 1.1], the set $X(\mu,b)_K$ is non-empty if and only if $[b] \in B(G,\{\mu\})$. Moreover [18, Theorem 1.1] says that for $J \subset K$ another σ-stable type, the natural projection $G(\breve{\mathbb{Q}}_p)/\mathcal{G}_J(\breve{\mathbb{Z}}_p) \to G(\breve{\mathbb{Q}}_p)/\mathcal{G}_K(\breve{\mathbb{Z}}_p)$ induces a $J_b(\mathbb{Q}_p)$-equivariant surjection

$$X(\mu,b)_J \to X(\mu,b)_K.$$

We will soon see that $X(\mu,b)_K$ can be identified with the set of $\overline{\mathbb{F}}_p$-points of a perfect scheme, which we will also denote by $X(\mu,b)_K$.

2.4.3. Let K be a σ-stable type, let $b \in G(\breve{\mathbb{Q}}_p)$ and consider the functor $X(\mu,b)'_K$ on $\mathbf{Aff}_k^{\mathrm{perf}}$ sending R to the set of isomorphism classes of commutative diagrams of modifications of $\mathcal{G}_K$-torsors on D_R

$$\begin{array}{ccc} \sigma^*\mathcal{E}_1 & \overset{\beta_1}{\dashrightarrow} & \mathcal{E}_1 \\ \downarrow{\scriptstyle \sigma^*\beta_0} & & \downarrow{\scriptstyle \beta_0} \\ \sigma^*\mathcal{E}^0 & \overset{b}{\dashrightarrow} & \mathcal{E}^0, \end{array} \tag{2.4.1}$$

such that $\beta_1 : \sigma^*\mathcal{E}_1 \dashrightarrow \mathcal{E}_1$, considered as an element of $\mathrm{Hk}_K(R)$, lies in $\cup_{w\in \mathrm{Adm}(\{\mu\})_K} \mathrm{Hk}_K(w)(R)$. Here b is the modification of the trivial G_K-torsor $\sigma^*\mathcal{E}^0 \simeq \mathcal{E}^0$ given by multiplication by b. We will sometimes refer to β_0 as a *quasi-isogeny* of shtukas from $(\mathcal{E}_1,\beta_1) \to (\mathcal{E}^0, b)$.

Lemma 2.4.4. *The morphism $X(\mu,b)'_K \to \mathrm{Gr}_K$ that sends a diagram as in* (2.4.1) *to $\beta_0 : \mathcal{E}_1 \dashrightarrow \mathcal{E}^0$, is a closed immersion. Moreover it identifies*

$$X(\mu,b)'_K(\overline{\mathbb{F}}_p) \subset \mathrm{Gr}_K(\overline{\mathbb{F}}_p) = G(\breve{\mathbb{Q}}_p)/\mathcal{G}_K(\breve{\mathbb{Z}}_p)$$

with the affine Deligne–Lusztig set $X(\mu,b)_K$ from Section 2.4.2.

Proof. Consider the functor $X(b)$ sending R to the set of isomorphism classes of commutative diagrams of modifications of $\mathcal{G}_K$-torsors on D_R

$$\begin{array}{ccc} \sigma^*\mathcal{E}_1 & \overset{\beta_1}{\dashrightarrow} & \mathcal{E}_1 \\ \downarrow{\scriptstyle \sigma^*\beta_0} & & \downarrow{\scriptstyle \beta_0} \\ \sigma^*\mathcal{E}^0 & \overset{b}{\dashrightarrow} & \mathcal{E}^0 \end{array} \tag{2.4.2}$$

as before, but now *without* the condition that $\beta_1 \in \bigcup_{w\in \mathrm{Adm}(\{\mu\})_K} \mathrm{Hk}_K(w)(R)$. As before, [19, the discussion after Remark 3.5] tells us that $X(\mu,b)'_K$ is a closed subfunctor of $X(b)$, and the lemma would follow if we could show that the map

$$f : X(b) \to \mathrm{Gr}_K$$

sending a diagram as in (2.4.2) to $\beta_0 : \mathcal{E}_1 \dashrightarrow \mathcal{E}^0$ is an isomorphism. The map f is an isomorphism because the map $g : \mathrm{Gr}_K \to X(b)$ sending $\beta_0 : \mathcal{E}_1 \dashrightarrow \mathcal{E}^0$ to the diagram

$$\begin{array}{ccc} \sigma^*\mathcal{E}_1 & \overset{\beta_1}{\dashrightarrow} & \mathcal{E}_1 \\ \downarrow{\scriptstyle \sigma^*\beta_0} & & \downarrow{\scriptstyle \beta_0} \\ \sigma^*\mathcal{E}^0 & \overset{b}{\dashrightarrow} & \mathcal{E}^0. \end{array}$$

with $\beta_1 = \beta_0^{-1} b \sigma^*\beta_0$ is an inverse to f.

We see that $X(\mu,b)'_K(\overline{\mathbb{F}}_p)$ is cut out from $X(b)(\overline{\mathbb{F}}_p) = G(\breve{\mathbb{Q}}_p)/\mathcal{G}_K(\breve{\mathbb{Z}}_p)$ by the condition that $\beta_1 \in \bigcup_{w\in \mathrm{Adm}(\{\mu\})_K} \mathrm{Hk}_K(w)(\overline{\mathbb{F}}_p)$, in other words, that

$$\beta_0^{-1} b \sigma^*\beta_0 \in \bigcup_{w\in \mathrm{Adm}(\{\mu\})} \mathcal{G}_K(\breve{\mathbb{Z}}_p) \dot{w} \mathcal{G}_K(\breve{\mathbb{Z}}_p).$$

This is precisely the condition cutting out $X(\mu,b)_K \subset G(\breve{\mathbb{Q}}_p)/\mathcal{G}_K(\breve{\mathbb{Z}}_p)$, and so we are done. □

From now on, we will write $X(\mu,b)_K$ for $X(\mu,b)'_K$ by abuse of notation. It follows from [15, Lemma 1.1] and [56, Corollary 2.5.3] that $X(\mu,b)_K$ is actually a perfect scheme that is perfectly locally of finite type.

If b' is σ-conjugate to b, that is if $b' = g^{-1}b\sigma(g)$ with $g \in G(\breve{\mathbb{Q}}_p)$, then $X(\mu,b)_K \simeq X(\mu,b')_K$ via the map

$$\begin{array}{ccc} \sigma^*\mathcal{E}_1 & \xrightarrow{\beta_1} & \mathcal{E}_1 \\ \downarrow{\scriptstyle \sigma^*\beta_0} & & \downarrow{\scriptstyle \beta_0} \\ \sigma^*\mathcal{E}^0 & \xrightarrow{b} & \mathcal{E}^0. \end{array} \mapsto \begin{array}{ccc} \sigma^*\mathcal{E}_1 & \xrightarrow{\beta_1} & \mathcal{E}_1 \\ \downarrow{\scriptstyle \sigma(g)\sigma^*\beta_0} & & \downarrow{\scriptstyle g\beta_0} \\ \sigma^*\mathcal{E}^0 & \xrightarrow{b'} & \mathcal{E}^0. \end{array}$$

We note that this map is nothing more than the action of $g \in LG(\overline{\mathbb{F}}_p)$ on $X(\mu,b)_K \subset \mathrm{Gr}_K$ via the natural left action of LG on Gr_K. For $b' = b$ this induces an action of the closed subgroup $F_b \subset LG$ on $X(\mu,b)_K$, where F_b is defined as the subfunctor of LG sending a perfect $\mathbb{F}_p$-algebra R to the group

$$F_b(R) = \{g \in LG(R) \mid g^{-1}b\sigma(g) = b\}.$$

The $\overline{\mathbb{F}}_p$-points of F_b are in bijection with $J_b(\mathbb{Q}_p)$, where J_b is the algebraic group over $\mathbb{Q}_p$ introduced in Section 2.4.2.

Lemma 2.4.5. *Consider the morphism $\Theta_b : X(\mu,b)_K \to \mathrm{Sht}_{G,K,\{\mu\},[b]}$, which sends a diagram as in (2.4.1) to $(\mathcal{E}_1,\beta_1)$. This morphism is F_b-invariant and induces an isomorphism of stacks in the pro-étale topology*

$$\mathrm{Sht}_{G,K,\{\mu\},[b]} \simeq [F_b\backslash X(\mu,b)_K]\,.$$

Moreover F_b is isomorphic to the locally profinite group scheme $\underline{J_b(\mathbb{Q}_p)}$ associated to the topological group $J_b(\mathbb{Q}_p)$.[8]

Proof. The morphism Θ_b is F_b-invariant, since the action of F_b on $X(\mu,b)_K$ doesn't change $(\mathcal{E}_1,\beta_1)$. For a scheme $T \mapsto \mathrm{Sht}_{G,K,\{\mu\},[b]}$, the set $X(\mu,b)_K(T)$ is the set of quasi-isogenies from $(\mathcal{E}_1,\beta_1)$ to $(\mathcal{E}^0_T, b_T)$, which is either empty or has a simply transitive action of the group $F_b(T)$ of self quasi-isogenies of $(\mathcal{E}^0_T, b_T)$. In other words, we have shown that Θ_b is a quasi-torsor for F_b. By [9, Theorem I.2.1], for any $\mathcal{G}_K$-shtuka $(\mathcal{E}_1,\beta_1) \in \mathrm{Sht}_{G,K,[b]}(T)$, the quasi-torsor of quasi-isogenies to $(\mathcal{E}^0_T, b_T)$ has a section pro-étale locally on T. Thus the map Θ_b is a pro-étale torsor for F_b, and so

$$\mathrm{Sht}_{G,K,\{\mu\},[b]} \simeq [F_b\backslash X(\mu,b)_K]\,.$$

It also follows from [9, Theorem I.2.1] that F_b is isomorphic to the locally profinite group scheme $\underline{J_b(\mathbb{Q}_p)}$ associated to $J_b(\mathbb{Q}_p)$. □

Corollary 2.4.6. *The stack $\mathrm{Sht}_{G,K,\{\mu\},[b]}$ is non-empty if and only if $[b] \in B(G,\{\mu\})$.*

Proof. This is a direct consequence of Lemma 2.4.5 in combination with the analogous result for $X(\mu,b)_K$, which is [18, Theorem 1.1]. □

3. Uniformisation of isogeny classes

In this section we will recall the construction of the Kisin–Pappas integral models of Shimura varieties of Hodge type with parahoric level structure, and recall the construction of Hamacher–Kim of shtukas on the perfections of their special fibers. We also discuss the change-of-parahoric maps constructed by Zhou in [55], and show that the shtukas of Hamacher–Kim are compatible with these maps using results of [43].

We then recall the results from Appendix A about the existence of CM lifts for Shimura varieties with very special level, and use that to deduce the existence of CM lifts for arbitrary parahorics. Next,

[8] For a topological group B we define $\underline{B}$ as the functor on $\mathbf{Aff}_k^{\mathrm{perf}}$ sending R to the group of continuous functions $|\operatorname{Spec} R| \to B$, where $|\operatorname{Spec} R|$ is the underlying topological space of $\operatorname{Spec} R$. When B is profinite this is representable by an affine group scheme, and thus when B is locally profinite it is representable by a group scheme.

we study how uniformisation 'lifts' along the change-of-parahoric maps. Concretely, we will show that uniformisation of isogeny classes at Iwahori level follows from uniformisation at very special level if a certain diagram of stacks on $\mathbf{Aff}_k^{\mathrm{perf}}$ is Cartesian.

3.1. **Integral models of Shimura varieties.** We recall the construction of the integral models of Shimura varieties of Hodge type in [25]. Let (G, X) be a Shimura datum with reflex field E and let $\{\mu_h\}$ be the $G(\mathbb{C})$-conjugacy class of cocharacters of $G_{\mathbb{C}}$ defined in [55, Section 6]. Let $\mathbb{A}_f$ denote the ring of finite adeles and $\mathbb{A}_f^p$ the subring of $\mathbb{A}_f$ with trivial p-component. Let $U_p \subset G(\mathbb{Q}_p)$ and $U^p \subset G(\mathbb{A}_f^p)$ be compact open subgroups, write $U = U^p U_p$. Then for U^p sufficiently small

$$G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/U$$

has the structure of an algebraic variety over $\mathbb{C}$, which has a canonical model $\mathbf{Sh}_U(G, X)$ over the reflex field E of (G, X). We will also consider the pro-varieties

$$\mathbf{Sh}_{U_p}(G, X) := \varprojlim_{U^p} \mathbf{Sh}_{U^p U_p}(G, X)$$
$$\mathbf{Sh}(G, X) := \varprojlim_{U} \mathbf{Sh}_U(G, X).$$

3.1.1. Let V be a vector space over $\mathbb{Q}$ of dimension $2g$ equipped with an alternating bilinear form ψ. For a $\mathbb{Q}$-algebra R, we write $V_R = V \otimes_{\mathbb{Q}} R$. Let G_V denote the corresponding group of symplectic similitudes and let $\mathcal{H}_V$ denote the set of homomorphisms $h : \mathbb{S} \to G_{V,\mathbb{R}}$ corresponding to the Siegel upper and lower half space, where $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ is the Deligne torus.

For the rest of this section, we fix an embedding of Shimura data $\iota : (G, X) \to (G_V, \mathcal{H}_V)$. We sometimes write G for $G_{\mathbb{Q}_p}$ when there is no risk of confusion. We will also assume for the rest of this section that the following conditions hold

$$G \text{ splits over a tamely ramified extension of } \mathbb{Q}_p \text{ and } p \nmid |\pi_1(G^{\mathrm{der}})|.$$

Let $\mathcal{G}$ be a connected parahoric subgroup of G, that is, $\mathcal{G} = \mathcal{G}_x = \mathcal{G}_x^\circ$ for some $x \in B(G, \mathbb{Q}_p)$, see Section 2.2.1. We will follow the notation of Section 2 to write $\mathcal{G} = \mathcal{G}_K$ for some σ-stable type $K \subset \mathbb{S}$. By [25, 2.3.15], after replacing ι by another symplectic embedding, there is a closed immersion $\mathcal{G}_K \to \mathcal{P}$, where $\mathcal{P}$ is a parahoric group scheme of G_V corresponding to the stabiliser of a lattice $V_{\mathbb{Z}_p} \subset V$. Upon scaling $V_{\mathbb{Z}_p}$, we may assume $V_{\mathbb{Z}_p}^\vee \subset V_{\mathbb{Z}_p}$. This induces a closed immersion of local models

$$M^{\mathrm{loc}}_{\mathcal{G}_K, X} \to M^{\mathrm{loc}}_{\mathcal{P}, \mathcal{H}_V} \otimes \mathcal{O}_{E,v}$$

for every place v of E above p. Here the local models are as introduced in [55, Section 3].

3.1.2. Let $U_V^p \subset G_V(\mathbb{A}_f^p)$ be a sufficiently small compact open subgroup. Let $V_{\mathbb{Z}_{(p)}} = V_{\mathbb{Z}_p} \cap V$ and write $G_{\mathbb{Z}_{(p)}}$ for the Zariski closure of G in $GL(V_{\mathbb{Z}_{(p)}})$, then $G_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p \cong \mathcal{G}_K$. The choice of $V_{\mathbb{Z}_{(p)}}$ gives rise to a compact open subgroup $U_{V,p} \subset G_V(\mathbb{Q}_p)$ which gives the Shimura variety $\mathbf{Sh}_{U_V}(G_V, \mathcal{H}_V)$ of level $U_V = U_V^p U_{V,p}$ an interpretation as a moduli space of (weakly polarised) abelian varieties up to prime-to-p isogeny, and hence an integral model $\mathscr{S}_{U_V}(G_V, \mathcal{H}_V)$ over $\mathbb{Z}_{(p)}$, which is described in [55, Section 6.3].

3.1.3. For the rest of this paper, we fix an algebraic closure $\overline{\mathbb{Q}}$ of E, and for each place v of $\mathbb{Q}$ an algebraic closure $\overline{\mathbb{Q}}_v$ together with an embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_v$. By [26, Lemma 2.1.2], we can choose U_V^p such that ι induces a closed immersion

$$\mathbf{Sh}_U(G, X) \hookrightarrow \mathbf{Sh}_{U_V}(G_V, \mathcal{H}_V)_E$$

defined over E. The choice of embedding $E \to \overline{\mathbb{Q}}_p$ determines a place v of E. Write $\mathcal{O}_{E,(v)}$ for the localisation of $\mathcal{O}_E$ at v, let E_v be the completion of E at v and $\mathcal{O}_{E,v}$ the ring of integers of E_v.

We assume the residue field has $q = p^r$ elements and as before k will denote an algebraic closure of $\mathbb{F}_q$. We define $\mathscr{S}_U(G,X)^-$ to be the Zariski closure of $\mathbf{Sh}_U(G,X)$ inside $\mathscr{S}_{U_V}(G_V,\mathcal{H}_V)\otimes_{\mathbb{Z}_{(p)}}\mathcal{O}_{E,(v)}$, and $\mathscr{S}_U(G,X)$ to be its normalisation. By construction, for $U_1^p \subset U_2^p$ compact open subgroups of $G(\mathbb{A}_f^p)$, there are finite étale transition maps $\mathscr{S}_{U_1^pU_p}(G,X) \to \mathscr{S}_{U_2^pU_p}(G,X)$ and we write $\mathscr{S}_{U_p}(G,X) := \varprojlim_{U^p} \mathscr{S}_{U^pU_p}(G,X)$. Under these assumptions we have the following result:

Theorem 3.1.4 ([25] Theorem 4.2.2, Theorem 4.2.7)**.** *The $\mathcal{O}_{E,(v)}$ scheme $\mathscr{S}_{U_p}(G,X)$ is a flat $G(\mathbb{A}_f^p)$-equivariant extension of $\mathbf{Sh}_{U_p}(G,X)$. Moreover $\mathscr{S}_U(G,X)_{\mathcal{O}_{E,v}}$ fits in a local model diagram*

$$\begin{array}{ccc} & \widetilde{\mathscr{S}_U}(G,X)_{\mathcal{O}_{E,v}} & \\ {\scriptstyle q}\swarrow & & \searrow{\scriptstyle \pi} \\ \mathscr{S}_U(G,X)_{\mathcal{O}_{E,v}} & & M^{\mathrm{loc}}_{\mathcal{G}_K,X} \end{array}$$

where q is a $\mathcal{G}_K$-torsor and π is smooth of relative dimension $\dim G$.

Note that the main result of [42] tells us that the integral model $\mathscr{S}_U(G,X)$ does not depend on the choice of Hodge embedding.

3.1.5. By [26, 1.3.2], the subgroup $G_{\mathbb{Z}_{(p)}}$ is the stabiliser of a collection of tensors $s_\alpha \in V^{\otimes}_{\mathbb{Z}_{(p)}}$. Let $h : \mathcal{A} \to \mathscr{S}_U(G,X)$ denote the pullback of the universal abelian variety on $\mathscr{S}_{U_V}(G_V,\mathcal{H}_V)$ and let $V_B := R^1h_{an,*}\mathbb{Z}_{(p)}$, where h_{an} is the map of complex analytic spaces associated to h. We also let $\mathcal{V} = R^1h_*\Omega^\bullet$ be the relative de Rham cohomology of $\mathcal{A}$. Using the de Rham isomorphism, the tensors s_α give rise to a collection of Hodge cycles $s_{\alpha,dR} \in \mathcal{V}_{\mathbb{C}}^{\otimes}$, where $\mathcal{V}_{\mathbb{C}}$ is the complex analytic vector bundle associated to $\mathcal{V}$. By [26, §2.2], these tensors are defined over E, and in fact over $\mathcal{O}_{E,(v)}$ by [25, Proposition 4.2.6].

Similarly, for a finite prime $\ell \neq p$, we let $\mathcal{V}_\ell = R^1h_{\acute{e}t*}\mathbb{Q}_\ell$ and $\mathcal{V}_p = R^1h_{\eta,\acute{e}t*}\mathbb{Z}_p$ where h_η is the generic fibre of h. Using the étale-Betti comparison isomorphism, we obtain tensors $s_{\alpha,\ell} \in \mathcal{V}_\ell^{\otimes}$ and $s_{\alpha,p} \in \mathcal{V}_p^{\otimes}$. For $* = B, dR, \ell$ and $x \in \mathscr{S}_{U^pU_p}(G,X)(T)$ for some $\mathcal{O}_{E,(v)}$-scheme T, we write $\mathcal{A}_x$ for the pullback of $\mathcal{A}$ to T via x and $s_{\alpha,*,x}$ for the pullback of $s_{\alpha,*}$ to T via x.

The image of x under $\mathscr{S}_{U^pU_p}(G,X) \to \mathscr{S}_{U_V}(G_V,\mathcal{H}_V)\otimes_{\mathbb{Z}_{(p)}}\mathcal{O}_{E,(v)}$ gives us a triple $(\mathcal{A}_x,\lambda,\epsilon^p_{U_V})$ where $(\mathcal{A}_x,\lambda)$ is a weakly polarised abelian variety up to prime-to-p isogeny and $\epsilon^p_{U_V}$ is a U_V^p level structure. As in [26, 3.4.2], the level structure $\epsilon^p_{U_V}$ can be promoted to a section:

$$\epsilon^p_U \in \Gamma(T, \underline{Isom}_{\lambda,\psi}(\widehat{V}^p(\mathcal{A}), V_{\mathbb{A}_f^p})/U^p) \tag{3.1.1}$$

which takes $s_{\alpha,\ell}$ to s_α for $\ell \neq p$.

3.1.6. Recall that $\overline{\mathbb{F}}_p$ is an algebraic closure of $\mathbb{F}_q$ and $\breve{\mathbb{Q}}_p = W(\overline{\mathbb{F}}_p)[1/p]$. Let $\overline{x} \in \mathscr{S}_U(G,X)(\overline{\mathbb{F}}_p)$ and $x \in \mathscr{S}_U(G,X)(\mathcal{O}_L)$ a point lifting $\overline{x}$, where $L/\breve{\mathbb{Q}}_p$ is a finite extension.

Let $\mathscr{G}_x$ denote the p-divisible group associated to $\mathcal{A}_x$ and $\mathscr{G}_{x,0}$ its special fiber. Then $T_p\mathscr{G}_x^\vee$ is identified with $H^1_{\acute{e}t}(\mathcal{A}_x,\mathbb{Z}_p)$ and we obtain Γ_K-invariant tensors $s_{\alpha,\acute{e}t,x} \in T_p\mathscr{G}^{\vee\otimes}$ whose stabiliser can be identified with $\mathcal{G}_K$. Let $\mathbb{D}_x := \mathbb{D}(\mathscr{G}_{x,0})$ be the contravariant Dieudonné module associated to the p-divisible group $\mathscr{G}_{x,0}$. We may apply the constructions of [55, Section 3] to obtain φ-invariant tensors $s_{\alpha,0,x} \in \mathbb{D}_x$, whose stabiliser group can be identified with $\mathcal{G}_K \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p$.

This means that we can upgrade the Dieudonné module of A_x to a $\mathcal{G}_K$-shtuka over $\overline{\mathbb{F}}_p$, and this gives a map (see [55, Proof of Axiom 4 in Section 8])

$$\mathscr{S}_U(G,X)(\overline{\mathbb{F}}_p) \to \mathrm{Sht}_{G,K,\{\mu\}}(\overline{\mathbb{F}}_p), \tag{3.1.2}$$

where $\{\mu\} = \{\sigma(\mu_h^{-1})\}$. It is a result of Hamacher–Kim ([14, Proposition 1], see [50, Poposition 4.4.1]) that there is a morphism $\mathrm{Sh}_{G,K,U^p} \to \mathrm{Sht}_{G,K,\{\mu\}}$ inducing (3.1.2) on $\overline{\mathbb{F}}_p$-points, where Sh_{G,K,U^p} is the perfection of the basechange to k of $\mathscr{S}_U(G,X)$.[9] It follows from [43, Theorem 1.3.4][10] that this morphism does not depend on the choice of Hodge embedding and moreover can be upgraded to a $G(\mathbb{A}_f^p)$-equivariant morphism

$$\mathrm{Sh}_{G,K} := \varprojlim_{U^p} \mathrm{Sh}_{G,K,U^p} \to \mathrm{Sht}_{G,K,\{\mu\}},$$

where $G(\mathbb{A}_f^p)$ acts trivially on $\mathrm{Sht}_{G,K,\{\mu\}}$.

It follows from [50, the discussion after Theorem 4.4.3] that the perfection of the special fiber of $M^{\mathrm{loc}}_{\mathcal{G}_K,X}$ can be identified with the closed subscheme $\mathrm{M}^{\mathrm{loc}}_{K,\{\mu\}}$ of the affine flag variety for $\mathcal{G}_K$ introduced in Section 2.2.13. Under this isomorphism, the right action of $L^+\mathcal{G}_K$ on $\mathrm{M}^{\mathrm{loc}}_{K,\{\mu\}}$, which factors through $\overline{\mathcal{G}}_K$, is identified with the $\overline{\mathcal{G}}_K$ action[11] on the perfection of $M^{\mathrm{loc}}_{G,K,\mu}$. Thus the local model diagram of Theorem 3.1.4 gives us a (perfectly smooth) morphism

$$\lambda_K : \mathrm{Sh}_{G,K,U^p} \to [\mathrm{M}^{\mathrm{loc}}_{K,\{\mu\}} / \overline{\mathcal{G}}_K].$$

3.1.7. Fix $n \geq 2$ and choose $m \gg 0$ such that the action $\mathrm{Ad}_\sigma L^+\mathcal{G}_K$ on $\mathrm{M}^{\mathrm{loc,n}}_{K,\{\mu\}}$ factors through $L^m\mathcal{G}_K$ and such that m satisfies the assumptions of Proposition 2.3.4. Let $\mathrm{Sht}^{(m,1)}_{G,K,\{\mu\}}$ be the stack of (m)-restricted shtukas of type $\{\mu\}$ from Section 2.3 and also consider the stack $\mathrm{Sht}^{(m,n),\mathrm{loc}}_{G,K,\{\mu\}}$ from Section 2.3.10. If we compose the morphism $\mathrm{Sh}_{G,K,U^p} \to \mathrm{Sht}_{G,K,\{\mu\}}$ constructed above with the natural map $\mathrm{Sht}_{G,K,\{\mu\}} \to \mathrm{Sht}^{(m,1)}_{G,K,\{\mu\}}$ we obtain a morphism

$$\mathrm{Sh}_{G,K,U^p} \to \mathrm{Sht}^{(m,1)}_{G,K,\{\mu\}}.$$

By [50, Theorem 4.4.3], the perfectly smooth map $\lambda_K : \mathrm{Sh}_{G,K,U^p} \to [\mathrm{M}^{\mathrm{loc}}_{K,\{\mu\}} / \overline{\mathcal{G}}_K]$ induced from the local model diagram fits in a commutative diagram

$$\begin{array}{ccc} \mathrm{Sh}_{G,K,U^p} & \longrightarrow & \mathrm{Sht}^{(m,1)}_{G,K,\{\mu\}} \\ & \searrow^{\lambda_K} & \downarrow \\ & & [\mathrm{M}^{\mathrm{loc}}_{\emptyset,\{\mu\}} / \overline{\mathcal{G}}_K], \end{array}$$

where $\mathrm{Sht}^{(m,1)}_{G,K,\{\mu\}} \to [\mathrm{M}^{\mathrm{loc}}_{K,\{\mu\}} / \overline{\mathcal{G}}_K]$ comes from the diagram in Remark 2.3.2. Moreover [50, Theorem 4.4.3] tells us that the map $\mathrm{Sh}_{G,K,U^p} \to \mathrm{Sht}^{(m,1)}_{G,K,\{\mu\}}$ is perfectly smooth.

Recall moreover that there is a natural map $\mathrm{Sht}_{G,K,\{\mu\}} \to \mathrm{Sht}^{(m,n),\mathrm{loc}}_{G,K,\{\mu\}}$ which induces a map $\mathrm{Sh}_{G,K,U^p} \to \mathrm{Sht}^{(m,n),\mathrm{loc}}_{G,K,\{\mu\}}$. When $\mathcal{G}_K$ is hyperspecial, then it is proved in [54, Proposition 7.2.4] that this map is perfectly smooth, and the proof adapts to the parahoric case as in the proof of [50, Theorem 4.4.3].

[9] The subscript K is used to signify the choice of σ-stable type $K \subset \mathbb{S}$ corresponding to the parahoric subgroup U_p.

[10] Pappas and Rapoport construct a 'local shtuka bounded by $\{\mu\}$' over the diamond associated to the formal completion of $\mathscr{S}_U(G,X)$ and prove uniqueness results for this object. By [43, Example 2.4.9], this induces a $\mathcal{G}_K$-shtuka over the perfect special fiber of $\mathscr{S}_U(G,X)$, which is of type $\{\mu\}$ by the discussion in [43, Section 2.4.3].

[11] Here we are writing $\overline{\mathcal{G}}_K$ for the perfection of the special fiber of $\mathcal{G}_K$, by abuse of notation.

Remark 3.1.8. It has been pointed out to us by Manuel Hoff and Xinwen Zhu that the proof of [54, Proposition 7.2.4] is not correct as written; the square in [54, top of page 113] does not commute. The same error seems to be present in the proof of [50, Theorem 4.4.3]. Fortunately, the perfect smoothness result that we will need is a consequence of the expected main result of the forthcoming PhD thesis of Manuel Hoff [22].

We can use the perfectly smooth map $\mathrm{Sh}_{G,K,U^p} \to \mathrm{Sht}^{(m,1)}_{G,K,\{\mu\}}$ to define the EKOR stratification on Sh_{G,K,U^p}, see Section 2.3.11. In particular for $w \in {}^K\mathrm{Adm}(\{\mu\})$ we will write $\mathrm{Sh}_{G,K,U^p}\{w\}$ for the locally closed EKOR stratum of Sh_{G,K,U^p}. Since $\mathrm{Sh}_{G,K,U^p} \to \mathrm{Sht}^{(m,1)}_{G,K,\{\mu\}}$ is perfectly smooth, we find that the closure of $\mathrm{Sh}_{G,K,U^p}\{w\}$ is given by

$$\mathrm{Sh}_{G,K,U^p}\{\preceq w\} := \bigcup_{w' \preceq w} \mathrm{Sh}_{G,K,U^p}\{w'\}$$

because the closure relations hold on $\mathrm{Sht}^{(m,1)}_{G,K,\{\mu\}}$.

3.1.9. *Isogeny classes.* Let $x \in \mathrm{Sh}_{G,K,U^p}(\overline{\mathbb{F}}_p)$, then attached to x is an abelian variety $\mathcal{A}_x$ over $\overline{\mathbb{F}}_p$. We write $\mathbb{D}_x$ for the contravariant Dieudonné module associated to the p-divisible group $\mathscr{G}_x$ of $\mathcal{A}_x$; then $\mathbb{D}_x$ is equipped with a corresponding set of tensors $s_{\alpha,0,x}$, see Section 3.1.6. Similarly, for $\ell \neq p$, the ℓ-adic Tate module $T_\ell\mathcal{A}_x$ is equipped with tensors $s_{\alpha,\ell,x} \in T_\ell\mathcal{A}_x^\otimes$.

Two points $x, x' \in \mathrm{Sh}_{G,K}(\overline{\mathbb{F}}_p)$ are said to lie in the same isogeny class if there exists a quasi-isogeny $\mathcal{A}_x \to \mathcal{A}_{x'}$ such that the induced maps $\mathbb{D}_x[1/p] \to \mathbb{D}_{x'}[1/p]$ and $V_\ell\mathcal{A}_x \otimes \mathbb{Q}_\ell \to V_\ell\mathcal{A}_{x'} \otimes \mathbb{Q}_\ell$ sends $s_{\alpha,0,x'}$ to $s_{\alpha,0,x}$ and $s_{\alpha,\ell,x}$ to $s_{\alpha,\ell,x'}$ for all $\ell \neq p$. We write $\mathscr{I}_x \subset \mathrm{Sh}_{G,K}(\overline{\mathbb{F}}_p)$ for the isogeny class of x.

For $x \in \mathrm{Sh}_{G,K}(\overline{\mathbb{F}}_p)$ we let I_x denote the reductive $\mathbb{Q}$-group associated to x as in [55, Section 9.2]; it is a subgroup of the algebraic group of self quasi-isogenies of the abelian variety A_x. It comes equipped with a natural map $I_{x,\mathbb{A}_f^p} \to G_{\mathbb{A}_f^p}$ coming from the tautological basis of the prime-to-p adelic Tate-module of A_x given by (the inverse limit over U^p of) (3.1.1). If we choose an isomorphism $\alpha : \mathbb{D}_x \simeq V_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p$ sending $s_{\alpha,0,x}$ to $s_\alpha \otimes 1$, under which the Frobenius of $\mathbb{D}_x$ corresponds to $b \in G(\breve{\mathbb{Q}}_p)$, then there is also an induced map $I_{x,\mathbb{Q}_p} \to J_b$. Note that an isomorphism α as above always exists, by [55, Section 5.6].

3.1.10. *Change of parahoric.* Now let $J \subset K$ be another σ-stable type, let $\mathcal{G}_J(\mathbb{Z}_p) =: U'_p \subset U_p$ and let $U' = U^pU'_p$. We will use Sh_{G,J,U^p} to denote the perfection of the special fiber of $\mathscr{S}_{U'}(G,X)$. By [55, Theorem 7.1], there is a proper morphism $\pi_{J,K} : \mathscr{S}_{U'}(G,X) \to \mathscr{S}_U(G,X)$ which induces the obvious forgetful morphism on generic fibers and induces a $G(\mathbb{A}_f^p)$-equivariant map

$$\varprojlim_{U^p} \mathscr{S}_{U'}(G,X) \to \varprojlim_{U^p} \mathscr{S}_U(G,X).$$

We now recall the construction of the forgetful map from [55, Section 7.2]: There are facets $\mathfrak{f}, \mathfrak{f}'$ of the extended Bruhat–Tits building $B(G,\mathbb{Q}_p)$ of $G_{\mathbb{Q}_p}$ such that U_p is the stabiliser of $\mathfrak{f}$ and such that U'_p is the stabiliser of $\mathfrak{f}'$. Under the embedding $\theta : B(G,\mathbb{Q}_p) \to B(G_V,\mathbb{Q}_p)$ induced by $G \to G_V$ we choose facets $\mathfrak{g}, \mathfrak{g}'$ containing $\theta(\mathfrak{f})$ and $\theta(\mathfrak{f}')$ respectively, and we let $M_p \subset G_V(\mathbb{Q}_p)$ be the stabiliser of $\mathfrak{g}$ and $M'_p \subset G_V(\mathbb{Q}_p)$ be the stabiliser of $\mathfrak{g}'$. As in [55, Section 8.1], the facets $\mathfrak{g}, \mathfrak{g}'$ correspond to lattice chains $\mathcal{L}$ and $\mathcal{L}'$ in $V_{\mathbb{Q}_p}$ respectively, with $\mathcal{L}'$ a refinement[12] of $\mathcal{L}$; note that [55] writes $\mathcal{L}'$ for what we call $\mathcal{L}$ and vice versa.

Then for sufficiently small $M^p \subset G_V(\mathbb{A}_f^p)$ there are moduli-theoretic integral models $\mathscr{S}_{M^pM'_p}(G_V,\mathcal{H}_V)$ and $\mathscr{S}_{M^pM_p}(G_V,\mathcal{H}_V)$ over $\mathbb{Z}_{(p)}$. The former is a moduli space of $\mathcal{L}'$-chains of (weakly polarised) abelian varieties up to prime-to-p isogeny with M^p level structure, as explained in [55, Proof of Axiom 1 in

[12]This means that $\mathcal{L}$ and $\mathcal{L}$' are chains of lattices in $V_{\mathbb{Q}_p}$ and that every lattice in $\mathcal{L}$ is also contained in $\mathcal{L}'$.

Section 8], and the latter is a moduli space of $\mathcal{L}$-chains of (weakly polarised) abelian varieties up to prime-to-p isogeny with M^p level structure. There is a natural proper forgetful map

$$\pi_{\mathcal{L}',\mathcal{L}} : \mathscr{S}_{M^pM'_p}(G_V, \mathcal{H}_V) \to \mathscr{S}_{M^pM_p}(G_V, \mathcal{H}_V)$$

which sends an $\mathcal{L}'$-chain of abelian varieties to the underlying $\mathcal{L}$-chain of abelian varieties.

Taking the direct sum of the lattices in the lattice chain $\mathcal{L}$ (resp. $\mathcal{L}'$) we get a symplectic space $V_{\mathcal{L}}$ (resp. $V_{\mathcal{L}'}$) and a lattice $V_{\mathcal{L},\mathbb{Z}_p}$ (resp. $V_{\mathcal{L}',\mathbb{Z}_p}$) in $V_{\mathcal{L},\mathbb{Q}_p}$ (resp. $V_{\mathcal{L}',\mathbb{Q}_p}$). Let us denote the stabiliser of $V_{\mathcal{L},\mathbb{Z}_p}$ (resp. $V_{\mathcal{L}',\mathbb{Z}_p}$) in $G_{V_{\mathcal{L}}}(\mathbb{Q}_p)$ (resp. $G_{V_{\mathcal{L}'}}(\mathbb{Q}_p)$) by J_p (resp. J'_p).

Then there are Hodge embeddings $(G_V, \mathcal{H}_V) \to (G_{V_{\mathcal{L}}}, \mathcal{H}_{V_{\mathcal{L}}})$ and $(G_V, \mathcal{H}_V) \to (G_{V_{\mathcal{L}'}}\mathcal{H}_{V_{\mathcal{L}'}})$, which take M_p to J_p and M'_p to J'_p respectively. These induce finite maps

$$\mathscr{S}_{M^pM'_p}(G_V, \mathcal{H}_V) \to \mathscr{S}_{J'^pJ'_p}(G_{V_{\mathcal{L}'}}, \mathcal{H}_{V_{\mathcal{L}'}}), \qquad \mathscr{S}_{M^pM_p}(G_V, \mathcal{H}_V) \to \mathscr{S}_{J^pJ_p}(G_{V_{\mathcal{L}}}, \mathcal{H}_{V_{\mathcal{L}}})$$

for some $J^p \subset G_{V_{\mathcal{L}}}(\mathbb{A}_f^p)$ and $J'^p \subset G_{V_{\mathcal{L}'}}(\mathbb{A}_f^p)$ sufficiently small, which take an $\mathcal{L}'$-set (resp. $\mathcal{L}$-set) of abelian varieties to the product of all the abelian varieties in the $\mathcal{L}'$-set (resp. the $\mathcal{L}$-set), equipped with the product polarisation and level structure. It is explained in [55, Equation 8.1 of Section 8] that our forgetful maps fit in a commutative diagram where all the horizontal maps are finite

$$\begin{array}{ccccc} \mathscr{S}_{U'}(G,X) & \longrightarrow & \mathscr{S}_{M^pM'_p}(G_V,\mathcal{H}_V)\otimes\mathcal{O}_{E,(v)} & \longrightarrow & \mathscr{S}_{J'^pJ'_p}(G_{V_{\mathcal{L}'}}\mathcal{H}_{V_{\mathcal{L}'}})\otimes\mathcal{O}_{E,(v)} \\ \downarrow{\scriptstyle \pi_{J,K}} & & \downarrow{\scriptstyle \pi_{\mathcal{L}',\mathcal{L}}} & & \\ \mathscr{S}_{U}(G,X) & \longrightarrow & \mathscr{S}_{M^pM_p}(G_V,\mathcal{H}_V)\otimes\mathcal{O}_{E,(v)} & \longrightarrow & \mathscr{S}_{J^pJ_p}(G_{V_{\mathcal{L}}},\mathcal{H}_{V_{\mathcal{L}}})\otimes\mathcal{O}_{E,(v)}. \end{array} \tag{3.1.3}$$

3.1.11. *Change of parahoric and isogeny classes.* We set $\mathrm{Sh}_{G,J} := \varprojlim_{U^p} \mathrm{Sh}_{G,J,U^p}$ and we let $\pi : \mathrm{Sh}_{G,J} \to \mathrm{Sh}_{G,K}$ denote the $G(\mathbb{A}_f^p)$-equivariant map induced by $\pi_{J,K}$. We now *define* isogeny classes in $\mathrm{Sh}_{G,J}(\overline{\mathbb{F}}_p)$ using the Hodge embedding $(G,X) \to (G_{V_{\mathcal{L}'}}\mathcal{H}_{V_{\mathcal{L}'}})$, as in Section 3.1.9. Similarly, we *define* isogeny classes in $\mathrm{Sh}_{G,K}(\overline{\mathbb{F}}_p)$ using the Hodge embedding $(G,X) \to (G_{V_{\mathcal{L}}}, \mathcal{H}_{V_{\mathcal{L}}})$. By [55, Proposition 7.7], the forgetful map is compatible with isogeny classes in the sense that for $x \in \mathrm{Sh}_{G,J}(\overline{\mathbb{F}}_p)$ we have $\pi(\mathscr{I}_x) \subset \mathscr{I}_{\pi(x)}$. We will need the following refinement:

Proposition 3.1.12. *Let $z, y \in \mathrm{Sh}_{G,J}(\overline{\mathbb{F}}_p)$ with the same image $x \in \mathrm{Sh}_{G,K}(\overline{\mathbb{F}}_p)$. Then z and y lie in the same isogeny class. In particular $\mathscr{I}_z = \pi^{-1}(\mathscr{I}_x)$.*

Proof. This follows as in the proof of [55, Proposition 7.7]. The point is that the two $\mathcal{L}'$-sets of abelian varieties associated to z and y have the same underlying $\mathcal{L}$-set of abelian varieties, namely the one associated to $\pi(z) = \pi(y)$. Therefore there is a unique quasi-isogeny of $\mathcal{L}'$-sets of abelian varieties extending the identification of the underlying $\mathcal{L}$-sets. This induces a quasi isogeny $A_x \dashrightarrow A_y$ on the products of the abelian varieties in the $\mathcal{L}'$-set, and it can be checked as in the proof of [55, Proposition 7.7] that this quasi-isogeny takes the tensors for A_y to the tensors for A_x. □

We will also need the following lemma.

Lemma 3.1.13. *The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Sh}_{G,J,U^p} & \longrightarrow & \mathrm{Sht}_{G,J,\{\mu\}} \\ \downarrow & & \downarrow \\ \mathrm{Sh}_{G,K,U^p} & \longrightarrow & \mathrm{Sht}_{G,K,\{\mu\}}\,. \end{array}$$

Proof. This is a consequence of [43, Corollary 4.3.2]. □

3.2. **CM Lifts.** In this section, we will prove a corollary of Theorem A.4.5. This corollary is a slight generalisation of Theorem 1.

Corollary 3.2.1. *Let $U_p' \subset G(\mathbb{Q}_p)$ be an arbitrary connected parahoric and suppose that there is a connected Iwahori subgroup U_p'' contained in U_p' and a connected very special parahoric subgroup U_p containing U_p''. Then each isogeny class of $\mathscr{S}_{U_p'}(G,X)(\overline{\mathbb{F}}_p)$ contains a point x which is the reduction of a special point on $\mathbf{Sh}_{U_p'}(G,X)$.*

Proof. Choose a connected Iwahori subgroup $U_p'' \subset U_p'$ and a connected very special parahoric subgroup $U_p \supset U_p''$ as in the assumptions of the theorem. We first prove the theorem for $\mathscr{S}_{U_p''}(G,X)$.

Let $z \in \mathscr{S}_{U_p''}(G,X)(\overline{\mathbb{F}}_p)$ and let x be its image in $\mathscr{S}_{U_p}(G,X)(\overline{\mathbb{F}}_p)$. Then the isogeny class $\mathscr{I}_x$ contains the reduction of a special point $P \in \mathscr{S}_{U_p}(G,X)(\overline{\mathbb{Q}})$ by Theorem A.4.5. This means that the Mumford–Tate group of P is a torus and the same is true for any lift $P'' \in \mathscr{S}_{U_p''}(G,X)(\overline{\mathbb{Q}})$. Thus we find that the inverse image of $\mathscr{I}_x$ under

$$\mathscr{S}_{U_p''}(G,X)(\overline{\mathbb{F}}_p) \to \mathscr{S}_{U_p}(G,X)(\overline{\mathbb{F}}_p)$$

contains the reduction of a special point. But by Proposition 3.1.12, this inverse image is equal to $\mathscr{I}_z$ and so every isogeny class in $\mathscr{S}_{U_p''}(G,X)(\overline{\mathbb{F}}_p)$ contains the reduction of a special point. A similar argument shows that every isogeny class in $\mathscr{S}_{U_p'}(G,X)(\overline{\mathbb{F}}_p)$ contains the reduction of a special point. □

3.3. **Lifting uniformisation.** From now on we let $K \subset \mathbb{S}$ be a σ-stable type corresponding to a connected very special parahoric. We let $U_p = \mathcal{G}_K(\mathbb{Z}_p)$ and $U_p' = \mathcal{G}_\emptyset(\mathbb{Z}_p)$; note that U_p' is a connected parahoric subgroup by Lemma 2.2.4. In this case, the commutative diagram from Lemma 3.1.13 is

$$\begin{array}{ccc} \mathrm{Sh}_{G,\emptyset,U^p} & \longrightarrow & \mathrm{Sht}_{G,\emptyset,\{\mu\}} \\ \downarrow & & \downarrow \\ \mathrm{Sh}_{G,K,U^p} & \longrightarrow & \mathrm{Sht}_{G,K,\{\mu\}}\,. \end{array} \tag{3.3.1}$$

The goal of this section is to prove the following result. Let $x \in \mathrm{Sh}_{G,\emptyset}(\overline{\mathbb{F}}_p)$ and choose an isomorphism $\mathbb{D}_x \simeq V_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p$ sending $s_{\alpha,0,x}$ to $s_\alpha \otimes 1$. Let $b \in G(\breve{\mathbb{Q}}_p)$ be the element corresponding to the Frobenius of $\mathbb{D}_x$ under this isomorphism.

Theorem 3.3.1. *If for every sufficiently small compact open subgroup U^p the diagram* (3.3.1) *is Cartesian, then for $z \in \mathscr{S}_{U_p'}(G,X)(\overline{\mathbb{F}}_p)$ with associated element $b \in G(\breve{\mathbb{Q}}_p)$ there is a $G(\mathbb{A}_f^p)$-equivariant bijection*

$$I_z(\mathbb{Q})\backslash X(\mu,b)_\emptyset(\overline{\mathbb{F}}_p) \times G(\mathbb{A}_f^p) \to \mathscr{I}_z.$$

3.3.2. Let $z \in \mathscr{S}_{U_p'}(G,X)(\overline{\mathbb{F}}_p)$ with image $x \in \mathscr{S}_{U_p}(G,X)(\overline{\mathbb{F}}_p)$ and let $b \in G(\breve{\mathbb{Q}}_p)$ be as in the statement of Theorem A.4.5. Then Theorem A.4.5 gives us a map of sets

$$G(\mathbb{A}_f^p) \times X(\mu,b)_K(\overline{\mathbb{F}}_p) \to \mathscr{I}_x$$

and Lemma 2.4.5 gives us a map of stacks $\Theta_b : X(\mu,b)_K \to \mathrm{Sht}_{G,K,\{\mu\},[b]}$.

Lemma 3.3.3. *The following diagram of groupoids commutes*

$$\begin{array}{ccc} G(\mathbb{A}_f^p) \times X(\mu,b)_K(\overline{\mathbb{F}}_p) & \xrightarrow{pr_2} & X(\mu,b)_K(\overline{\mathbb{F}}_p) \\ \downarrow & & \downarrow \Theta_b \\ \mathscr{I}_x & \longrightarrow & \mathrm{Sht}_{G,K,\{\mu\},[b]}(\overline{\mathbb{F}}_p). \end{array}$$

Proof. This follows from the compatibility of the uniformisation map with the 'joint stratification' $\mathrm{Sh}_{G,K,U^p}(\overline{\mathbb{F}}_p) \to \mathrm{Sht}_{G,K,\{\mu\}}(\overline{\mathbb{F}}_p)$, as discussed in the proof of Axiom 4(b) in Section 8 of [55]. □

Lemma 3.3.4. *Let $\mathscr{J}_x$ be the inverse image in $\mathrm{Sh}_{G,\emptyset}(\overline{\mathbb{F}}_p)$ of $\mathscr{I}_x \subset \mathrm{Sh}_{G,K}(\overline{\mathbb{F}}_p)$. If the assumption of Theorem 3.3.1 holds, then there is a $G(\mathbb{A}_f^p)$-equivariant bijection*

$$\mathscr{J}_x \simeq I_x(\mathbb{Q})\backslash X(\mu,b)_\emptyset(\overline{\mathbb{F}}_p) \times G(\mathbb{A}_f^p).$$

Proof. Taking the inverse limit over U^p of the Cartesian diagram of (3.3.1), we get the following $G(\mathbb{A}_f^p)$-equivariant Cartesian diagram of groupoids

$$\begin{array}{ccc} \mathscr{J}_x & \longrightarrow & \mathrm{Sht}_{G,\emptyset,\{\mu\}}(\overline{\mathbb{F}}_p) \\ \downarrow & & \downarrow \\ \mathscr{I}_x & \longrightarrow & \mathrm{Sht}_{G,K,\{\mu\}}(\overline{\mathbb{F}}_p). \end{array} \tag{3.3.2}$$

Theorem A.4.5 gives us a bijection $I_x(\mathbb{Q})\backslash X(\mu,b)_K(\overline{\mathbb{F}}_p) \times G(\mathbb{A}_f^p) \to \mathscr{I}_x$, and Lemma 2.4.5 gives us equivalences of groupoids

$$\mathrm{Sht}_{G,K,\{\mu\}}(\overline{\mathbb{F}}_p) \simeq \left[\frac{X(\mu,b)_K(\overline{\mathbb{F}}_p)}{J_b(\mathbb{Q}_p)}\right], \qquad \mathrm{Sht}_{G,\emptyset,\{\mu\}}(\overline{\mathbb{F}}_p) \simeq \left[\frac{X(\mu,b)_\emptyset(\overline{\mathbb{F}}_p)}{J_b(\mathbb{Q}_p)}\right].$$

Lemma 3.3.3 tells us that we can identify (3.3.2) with

$$\begin{array}{ccc} \mathscr{J}_x & \longrightarrow & \left[\frac{X(\mu,b)_\emptyset(\overline{\mathbb{F}}_p)}{J_b(\mathbb{Q}_p)}\right] \\ \downarrow & & \downarrow \\ I_x(\mathbb{Q})\backslash X(\mu,b)_K(\overline{\mathbb{F}}_p) \times G(\mathbb{A}_f^p) & \longrightarrow & \left[\frac{X(\mu,b)_K(\overline{\mathbb{F}}_p)}{J_b(\mathbb{Q}_p)}\right], \end{array}$$

such that the bottom map is induced by the projection map $X(\mu,b)_K(\overline{\mathbb{F}}_p) \times G(\mathbb{A}_f^p) \to X(\mu,b)_K(\overline{\mathbb{F}}_p)$ and the right vertical map is induced by the natural map $X(\mu,b)_\emptyset(\overline{\mathbb{F}}_p) \to X(\mu,b)_K(\overline{\mathbb{F}}_p)$. But now it is clear that there is a $G(\mathbb{A}_f^p)$-equivariant bijection

$$\mathscr{J}_x \simeq I_x(\mathbb{Q})\backslash X(\mu,b)_\emptyset(\overline{\mathbb{F}}_p) \times G(\mathbb{A}_f^p).$$

□

Proof of Theorem 3.3.1. The theorem is a direct consequence of Lemma 3.3.4, which proves uniformisation for $\mathscr{J}_x$, and Proposition 3.1.12, which proves that $\mathscr{J}_x = \mathscr{I}_z$. □

3.4. **Uniformisation and connected components.** Define $G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap G(\mathbb{R})_+$, with $G(\mathbb{R})_+$ the inverse image of the identity component (in the real topology) of $G^{\mathrm{ad}}(\mathbb{R})$ under the natural map $G(\mathbb{R}) \to G^{\mathrm{ad}}(\mathbb{R})$. Let $\rho : G^{\mathrm{sc}} \to G^{\mathrm{der}}$ be the simply-connected cover of the derived subgroup of G; we will sometimes conflate groups like $G^{\mathrm{sc}}(\mathbb{Q})$ and $G^{\mathrm{sc}}(\mathbb{A}_f^p)$ with their images under ρ by abuse of notation. Consider the set

$$\pi(G) := \varprojlim_{U^p} G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/U^pU_p.$$

Now we have $\rho(G^{\mathrm{sc}}(\mathbb{Q})) \subset G(\mathbb{Q})_+$ since $G^{\mathrm{sc}}(\mathbb{R})$ is connected, and strong approximation for G^{sc} away from ∞, see [44, Theorem 7.12], tells us that the closure of $\rho(G^{\mathrm{sc}}(\mathbb{Q}))$ in $G(\mathbb{A}_f)$ contains $\rho(G^{\mathrm{sc}}(\mathbb{A}_f))$. Moreover, the subset $G(\mathbb{Q})_+\rho(G^{\mathrm{sc}}(\mathbb{A}_f))$ is closed in $G(\mathbb{A}_f)$ since (G,X) is of Hodge type, see [7, Section

2.0.15]. This means that $G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/\rho(G^{\mathrm{sc}}(\mathbb{A}_f))$ is Hausdorff, and we can therefore deduce from [37, Lemma 4.20] that the natural map

$$G(\mathbb{Q})_+\backslash \frac{G(\mathbb{A}_f)}{\rho(G^{\mathrm{sc}}(\mathbb{A}_f))}/U_p \to \pi(G)$$

is a homeomorphism. We see that $\pi(G)$ is an abelian group since $\frac{G(\mathbb{A}_f)}{\rho(G^{\mathrm{sc}}(\mathbb{A}_f))}$ is.

3.4.1. By Lemma 3.4.2 below, we may make the identification

$$\pi(G) = G(\mathbb{Q})_+\backslash \left(\pi_1(G)_I^\sigma \times \frac{G(\mathbb{A}_f^p)}{\rho(G^{\mathrm{sc}}(\mathbb{A}_f^p))} \right).$$

In particular there is a natural surjective group homomorphism $\pi_1(G)_I^\sigma \times G(\mathbb{A}_f^p) \to \pi(G)$.

Lemma 3.4.2. *Let $\mathbb{G}$ be a connected reductive group over $\mathbb{Q}_p$ and let $\mathcal{G}$ be a parahoric group scheme for $\mathbb{G}$. Then there is a natural isomorphism*

$$\frac{\mathbb{G}(\mathbb{Q}_p)}{\rho(\mathbb{G}^{\mathrm{sc}}(\mathbb{Q}_p)) \cdot \mathcal{G}(\mathbb{Z}_p)} \simeq \pi_1(\mathbb{G})_I^\sigma.$$

Proof. Recall that we have the surjective Kottwitz homomorphism $\tilde{k}_{\mathbb{G}} : \mathbb{G}(\breve{\mathbb{Q}}_p) \to \pi_1(\mathbb{G})_I$ with kernel given by $\rho(\mathbb{G}^{\mathrm{sc}}(\breve{\mathbb{Q}}_p)) \cdot \mathcal{T}(\breve{\mathbb{Z}}_p) = \rho(\mathbb{G}^{\mathrm{sc}}(\breve{\mathbb{Q}}_p)) \cdot \mathcal{G}(\breve{\mathbb{Z}}_p)$ (see [41, Lemma 17 of the appendix]), where $\mathcal{T}$ is the connected Néron model of a standard torus T of $\mathbb{G}$. Recall moreover that $\tilde{k}_{\mathbb{G}}$ restricts to a surjective map $\mathbb{G}(\mathbb{Q}_p) \to \pi_1(\mathbb{G})_I^\sigma$ by [29, Section 7.7]. Thus when $\mathbb{G} = T$ is a torus, we have a short exact sequence

$$0 \to \mathcal{T}(\mathcal{O}_L) \to T(L) \to \pi_1(G)_I \to 0,$$

that remains exact upon taking σ-invariants, proving the lemma for tori. If $\mathbb{G}^{\mathrm{der}}$ is simply connected, then there is a canonical identification $\pi_1(\mathbb{G}) = \pi_1(\mathbb{G}^{\mathrm{ab}})$, where $\mathbb{G}^{\mathrm{ab}}$ is the maximal abelian quotient of $\mathbb{G}$. We can consider the morphism of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{G}^{\mathrm{der}}(\mathbb{Q}_p) & \longrightarrow & \mathbb{G}(\mathbb{Q}_p) & \longrightarrow & \mathbb{G}^{\mathrm{ab}}(\mathbb{Q}_p) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 1 & \longrightarrow & \pi_1(\mathbb{G})_I^\sigma & \longrightarrow & \pi_1(\mathbb{G}^{\mathrm{ab}})_I^\sigma & \longrightarrow & 1. \end{array}$$

The lemma now follows from the well known fact that the image of $\mathcal{G}(\mathbb{Z}_p)$ in $\mathbb{G}^{\mathrm{ab}}(\mathbb{Q}_p)$ is equal to $\mathcal{D}(\mathbb{Z}_p)$, where $\mathcal{D}$ is the connected Néron model of $\mathbb{G}^{\mathrm{ab}}$. For general $\mathbb{G}$, we can reduce to the case that $\mathbb{G}^{\mathrm{der}}$ is simply connected using a z-extension argument, see the proof of [41, Lemma 17 of the appendix] or [29, Section 7.7]. □

3.4.3. Define (cf. [7, Section 2.1.3])

$$\pi(G, X) := \varprojlim_{U^p} \pi_0(\mathbf{Sh}_U(G, X)_{\overline{\mathbb{C}}}) = \varprojlim_{U^p} G(\mathbb{Q})\backslash \left(\pi_0(X) \times G(\mathbb{A}_f)/U^p U_p\right).$$

This is a quotient of

$$\varprojlim_{U} G(\mathbb{Q})\backslash \left(\pi_0(X) \times G(\mathbb{A}_f)/U\right),$$

on which $G(\mathbb{A}_f)$ acts through the abelian group $G(\mathbb{A}_f)/\rho(G^{\mathrm{sc}}(\mathbb{A}_f))$, again by strong approximation for G^{sc} away from infinity. By the discussion above, this induces an action of $G(\mathbb{A}_f^p) \times \pi_1(G)_I^\sigma$ on $\pi(G, X)$ which makes it into a torsor for $\pi(G)$.

Recall that U_p is a very special parahoric, which implies that the integral model $\mathscr{S}_{U^pU_p}(G,X)$ has locally integral special fiber; see [25, Collary 4.6.26]. Then [36, Corollary 4.1.11] tells us that for all choices of U^p, for each finite extension F of the reflex field E and any place w of F extending v, the natural maps

$$\pi_0(\mathbf{Sh}_{U^pU_p}(G,X)\otimes_E F) \leftarrow \pi_0(\mathscr{S}_{U^pU_p}(G,X)\otimes_{\mathcal{O}_{E,(v)}}\mathcal{O}_{F,(w)}) \to \pi_0(\mathrm{Sh}_{G,K,U^p}\otimes k(w))$$

are isomorphisms. Thus there is a natural $G(\mathbb{A}_f^p)$-equivariant isomorphism $\pi_0(\mathrm{Sh}_{G,K}) \to \pi(G,X)$, which turns $\pi_0(\mathrm{Sh}_{G,K})$ into a torsor for $\pi(G)$ and equips it with an action of $G(\mathbb{A}_f^p)\times\pi_1(G)_I^\sigma$.

3.4.4. As before $\mathcal{G}_K$ denotes a connected very special parahoric group scheme. Let $x\in\mathrm{Sh}_{G,K}(\overline{\mathbb{F}}_p)$ and $b\in G(\breve{\mathbb{Q}}_p)$ the associated element that is well-defined up to $\mathcal{G}_K(\breve{\mathbb{Z}}_p)$-conjugacy. The Kottwitz homomorphism induces a natural map

$$\kappa : X(\mu,b)_K \to \mathrm{Gr}_K \to \pi_0(\mathrm{Gr}_K) \simeq \pi_1(G)_I,$$

with image $c_{[b],\mu}+\pi_1(G)_I^\sigma \subset \pi_1(G)_I$, see [19, Lemma 6.1]. As in [55, Section 6.7] we have $1\in X(\mu,b)_K(\overline{\mathbb{F}}_p)$, which implies that the coset $c_{[b],\mu}+\pi_1(G)_I^\sigma$ contains 1 and is thus equal to $\pi_1(G)_I^\sigma$. In particular, the map κ takes values in $\pi_1(G)_I^\sigma$. Theorem A.4.5 gives us a $G(\mathbb{A}_f^p)$-equivariant map

$$i_x : X(\mu,b)_K(\overline{\mathbb{F}}_p)\times G(\mathbb{A}_f^p)\to \mathrm{Sh}_{G,K}(\overline{\mathbb{F}}_p),$$

sending $(1,1)$ to x.

Proposition 3.4.5. *Consider the composition $X(\mu,b)_K(\overline{\mathbb{F}}_p)\times G(\mathbb{A}_f^p)\to \mathrm{Sh}_{G,K}(\overline{\mathbb{F}}_p)\to\pi_0(\mathrm{Sh}_{G,K}) = \pi(G,X)$, and let $\underline{x}$ be the image of x in $\pi(G,X)$. Then the image of (y,g^p) in $\pi(G,X)$ is given by*

$$(\kappa(y),g^p)\cdot\underline{x},$$

where $\cdot$ denotes the natural action of $\pi_1(G)_I^\sigma\times G(\mathbb{A}_f^p)$ on $\pi(G,X)$ constructed above.

Proof. By the $G(\mathbb{A}_f^p)$-equivariance of the map i_x, it suffices to prove the theorem for $g^p=1$ or for the map $X(\mu,b)_K(\overline{\mathbb{F}}_p)\to\mathrm{Sh}_{G,K}(\overline{\mathbb{F}}_p)$. The map $X(\mu,b)_K(\overline{\mathbb{F}}_p)\to\mathrm{Sh}_{G,K}(\overline{\mathbb{F}}_p)$ upgrades to a map of perfect schemes $X(\mu,b)_K\to\mathrm{Sh}_{G,K}$ by the proof of [20, Proposition 5.2.2]. Therefore the image of $y\in X(\mu,b)_K(\overline{\mathbb{F}}_p)$ in $\pi(G,X)$ only depends on the connected component that y lies in. Thus the result is true for a union of connected components $X(\mu,b)_K^\circ$ of $X(\mu,b)_K$. Moreover, the result is clearly true for $y=1$.

Now we follow the proof of Proposition A.4.3 and freely use the notation from that proof: Let $M\subset G_{\mathbb{Q}_p}$ be the standard Levi subgroup given by the centraliser of the Newton cocharacter $\overline{\nu}_b$. By Theorem A.1.3, there exists $\lambda\in I_{\mu,b,M}$ and an element

$$g\in X(\mu,b)_K^\circ\cap X^M(\lambda,b)_M.$$

We may then replace x by $i_x(g)$ to assume that $b\in M(\breve{\mathbb{Q}}_p)$ and furthermore that $b=\dot{\tau}_\lambda$ where $\tau_\lambda\in\Omega_M$ corresponds to $\kappa_M(b)\in\pi_1(M)_I$.

Arguing as in the proof of A.4.3, we can find a finite extension L of $\breve{\mathbb{Q}}_p$ and choose an (M,μ_y)-adapted lifting $\tilde{\mathscr{G}}/\mathcal{O}_L$ of $\mathscr{G}_x$ (cf. [55, Definition 4.6]) which corresponds to a point $\tilde{x}\in\mathscr{S}_{U_p}(G,X)(\mathcal{O}_L)$. The construction in [55, Proposition 5.14] gives us a map

$$\iota : M(\mathbb{Q}_p)/\mathcal{M}(\mathbb{Z}_p)\to X^M(\lambda,b)_{K_M},$$

whose composition with $X^M(\lambda,b)_{K_M} \to X(\mu,b)_K$ fits into the following commutative diagram

$$\begin{array}{ccc} M(\mathbb{Q}_p) & \longrightarrow & X(\mu,b)_K(\overline{\mathbb{F}}_p) \\ & \searrow \quad \swarrow_{\kappa} & \\ & \pi_1(G)_I^\sigma, & \end{array}$$

where the left diagonal map is the composition $M(\mathbb{Q}_p) \to G(\mathbb{Q}_p) \to \pi_1(G)_I^\sigma$. Choose a lift of $\tilde{x}$ to a point $z \in \mathbf{Sh}(G,X)(C)$, where C is an algebraic closure of $\breve{\mathbb{Q}}_p$. Then by construction the map ι fits into the following diagram (compare with the diagram in [27, Corollary 1.4.12])

$$\begin{array}{ccc} M(\mathbb{Q}_p) & \longrightarrow & \mathscr{S}_{U_p}(G,X)(\mathcal{O}_C) \\ \downarrow & & \downarrow \\ X(\mu,b)_K(\overline{\mathbb{F}}_p) & \longrightarrow & \mathrm{Sh}_{G,K}(\overline{\mathbb{F}}_p). \end{array}$$

Here the top horizontal map is given by the (Hecke) action of $M(\mathbb{Q}_p) \subset G(\mathbb{Q}_p)$ on $z \in \mathbf{Sh}(G,X)(C)$ followed by projection to back to $\mathscr{S}_{U_p}(C)$ and extending to $\mathscr{S}_{U_p}(G,X)(\mathcal{O}_C)$.

We see that elements $g \in X(\mu,b)_K(\overline{\mathbb{F}}_p)$ in the image of $M(\mathbb{Q}_p) \to X(\mu,b)_K(\overline{\mathbb{F}}_p)$ satisfy the conclusion of the proposition. Moreover, this means that the result holds for all points $g \in X(\mu,b)_K(\overline{\mathbb{F}}_p)$ lying in a connected component of $X(\mu,b)_K$ intersecting the image of the map $M(\mathbb{Q}_p) \to X(\mu,b)_K(\overline{\mathbb{F}}_p)$. But the map

$$M(\mathbb{Q}_p)/\mathcal{M}(\mathbb{Z}_p) \to \pi_0(X^M(\lambda,b)_{K_M})$$

is surjective by [55, Proposition 5.19] and moreover

$$\pi_0(X^M(\lambda,b)_{K_M}) \to \pi_0(X(\mu,b)_K)$$

is surjective by Theorem A.1.3. Thus every connected component of $X(\mu,b)_K$ contains a point in the image of $M(\mathbb{Q}_p) \to X(\mu,b)_K(\overline{\mathbb{F}}_p)$, and so we are done. □

Corollary 3.4.6. *Let $\tau \in \mathrm{Adm}(\{\mu\})$ be the unique element of length zero. Then*

$$\mathrm{Sh}_{G,\emptyset,U^p}(\tau) \to \pi_0(\mathrm{Sh}_{G,K,U^p})$$

is surjective.

Proof. It suffices to prove this for the analogous map $\mathrm{Sh}_{G,\emptyset}(\tau) \to \pi_0(\mathrm{Sh}_{G,K})$. Since $\mathrm{Sh}_{G,\emptyset}(\tau)$ is contained in the basic locus we can use [55, Proposition 6.5(i)] to produce for $x \in \mathrm{Sh}_{G,\emptyset}(\tau)$ a uniformisation map

$$X(\mu,b)_\emptyset \times G(\mathbb{A}_f^p) \to \mathrm{Sh}_{G,\emptyset}$$

which, as in [55, proof of Axiom 5], restricts to a map

$$i_x : X(\mu,b)_\emptyset(\tau) \times G(\mathbb{A}_f^p) \to \mathrm{Sh}_{G,\emptyset}(\tau).$$

Moreover, the following diagram commutes (by construction, see [55, Proposition 7.8])

$$\begin{array}{ccc} X(\mu,b)_\emptyset(\tau) \times G(\mathbb{A}_f^p) & \xrightarrow{i_x} & \mathrm{Sh}_{G,\emptyset}(\tau) \\ \downarrow & & \downarrow \\ X(\mu,b)_K \times G(\mathbb{A}_f^p) & \xrightarrow{i_{\pi_{\emptyset,K}(x)}} & \mathrm{Sh}_{G,K}\,. \end{array}$$

Since $X(\mu,b)_\emptyset(\tau) \subset X(\mu,b)_\emptyset$ is $J_b(\mathbb{Q}_p)$-stable, it follows that its image in $X(\mu,b)_K$ is $J_b(\mathbb{Q}_p)$-stable. Thus its image surjects onto $\pi_1(G)_I^\sigma$ via κ, because $J_b(\mathbb{Q}_p)$ surjects onto $\pi_1(G)_I^\sigma$ since b is basic, see

Lemma 3.4.2. The result now follows from Proposition 3.4.5 and the fact that $\pi_1(G)_I^\sigma \times G(\mathbb{A}_f^p)$ acts transitively on $\pi_0(\mathrm{Sh}_{G,K})$. □

Corollary 3.4.7. *For $w \in \mathrm{Adm}(\{\mu\})$, the map*

$$\mathrm{Sh}_{G,\emptyset,U^p}(w) \to \pi_0(\mathrm{Sh}_{G,K,U^p})$$

is surjective.

Proof. By [16, Theorem 4.1], this follows from Corollary 3.4.6. □

4. The Cartesian diagram

Let the notation be as in Section 3, in particular $\mathcal{G}_K$ is a connected very special parahoric group scheme. Define a sheaf $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}$ via the following fiber product diagram

$$\begin{array}{ccc} \widehat{\mathrm{Sh}}_{G,\emptyset,U^p} & \longrightarrow & \mathrm{Sht}_{G,\emptyset,\{\mu\}} \\ \downarrow & & \downarrow \\ \mathrm{Sh}_{G,K,U^p} & \longrightarrow & \mathrm{Sht}_{G,K,\{\mu\}}\,. \end{array} \tag{4.0.1}$$

In particular, $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}$ is $\mathrm{Sh}_{G,U',\star}$ from the introduction. Proposition 2.2.15 tells us that $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}$ is (representable by) a perfect algebraic space which is perfectly proper over Sh_{G,K,U^p}. The universal property of the fiber product gives us a morphism $\iota : \mathrm{Sh}_{G,\emptyset,U^p} \to \widehat{\mathrm{Sh}}_{G,\emptyset,U^p}$ and the goal of this section is to show that ι is an isomorphism, under some hypotheses.

In Section 4.1 we will show that ι is a closed immersion. In Section 4.2 we will show that $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}$ is equidimensional of the same dimension as Sh_{G,K,U^p}. In Section 4.3 we will show that each irreducible component of $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}$ can be moved into $\mathrm{Sh}_{G,\emptyset,U^p}$ using prime-to-p Hecke operators. We prove this by degenerating to the zero-dimensional KR stratum, which we describe explicitly using Rapoport–Zink uniformisation of the basic locus.

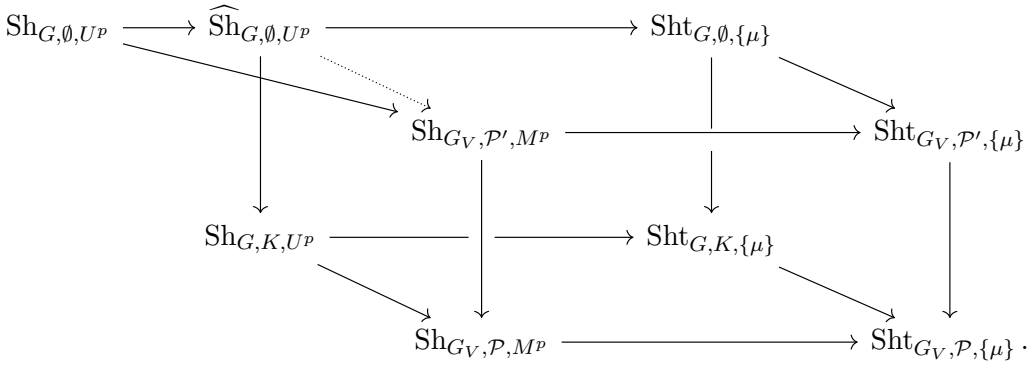
4.1. The natural map is a closed immersion.

Because the morphism $\mathrm{Sh}_{G,K} \to \mathrm{Sht}_{G,K,\{\mu\}}$ is $G(\mathbb{A}_f^p)$-equivariant, see [43, Theorem 1.3.4], we can form $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}$ for every choice of prime-to-p level subgroup U^p. Then there is an induced action of $G(\mathbb{A}_f^p)$ on $\widehat{\mathrm{Sh}}_{G,\emptyset} := \varprojlim_{U^p} \widehat{\mathrm{Sh}}_{G,\emptyset,U^p}$, such that the natural maps $\mathrm{Sh}_{G,\emptyset} \to \widehat{\mathrm{Sh}}_{G,\emptyset}$ and $\widehat{\mathrm{Sh}}_{G,\emptyset} \to \mathrm{Sh}_{G,K}$ are $G(\mathbb{A}_f^p)$-equivariant.

4.1.1. Let $\mathcal{P}, \mathcal{P}'$ be the parahoric group schemes with $\mathcal{P}(\mathbb{Z}_p) = M_p$ and $\mathcal{P}'(\mathbb{Z}_p) = M_p'$, see Section 3.1.10. Let $\mathrm{Sh}_{G_V,\mathcal{P}',M^p}$ and $\mathrm{Sh}_{G_V,\mathcal{P},M^p}$ be the perfections of the geometric special fibers of the schemes (introduced in Section 3.1.10)

$$\mathscr{S}_{M^pM_p'}(G_V,\mathcal{H}_V) \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{E,v} \quad \text{and} \quad \mathscr{S}_{M^pM_p}(G_V,\mathcal{H}_V) \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{E,v},$$

respectively. Now consider the following commutative diagram deduced from (3.1.3) (which is commutative by [43, Theorem 4.3.1]).



(4.1.1)

Lemma 4.1.2. *The front face of the cube, that is, the square involving* $\mathrm{Sh}_{G_V,\mathcal{P}',M^p}$, $\mathrm{Sht}_{G_V,\mathcal{P}',\{\mu\}}$, $\mathrm{Sh}_{G_V,\mathcal{P},M^p}$ *and* $\mathrm{Sht}_{G_V,\mathcal{P},\{\mu\}}$ *is Cartesian.*

Proof. The stack $\mathrm{Sht}_{G_V,\mathcal{P}',\{\mu\}}$ is[13]a moduli stack of $\mathcal{L}'$-chains of (polarised) p-divisible groups and the stack $\mathrm{Sht}_{G_V,\mathcal{P},\{\mu\}}$ is a moduli stack of $\mathcal{L}$-chains of polarised p-divisible groups. The natural map $\mathrm{Sh}_{G_V,\mathcal{P}',M^p} \to \mathrm{Sht}_{G_V,\mathcal{P}',\{\mu\}}$ sends an $\mathcal{L}'$-chain of abelian varieties to the corresponding $\mathcal{L}'$-chain of p-divisible groups. The map $\mathrm{Sh}_{G_V,\mathcal{P},M^p} \to \mathrm{Sht}_{G_V,\mathcal{P},\{\mu\}}$ has a similar description. Moreover, the map $\mathrm{Sht}_{G_V,\mathcal{P}',\{\mu\}} \to \mathrm{Sht}_{G_V,\mathcal{P},\{\mu\}}$ sends an $\mathcal{L}'$-chain of (polarised) p-divisible groups to the underlying $\mathcal{L}$-chain of (polarised) p-divisible groups.

The statement of the lemma now comes down to the following claim: Given an $\mathcal{L}$-chain $A_{\mathcal{L}}$ of (weakly polarised) abelian varieties, an $\mathcal{L}'$-chain $X_{\mathcal{L}'}$ of (polarised) p-divisible groups and an isomorphism from $A[p^\infty]_{\mathcal{L}}$ to the underlying $\mathcal{L}$-chain of $X_{\mathcal{L}}$, then there is a unique $\mathcal{L}'$-chain of abelian varieties $A_{\mathcal{L}'}$ with underlying $\mathcal{L}$-chain given by $A_{\mathcal{L}}$ and with p-divisible group $A[p^\infty]_{\mathcal{L}'} = X_{\mathcal{L}'}$. This claim follows from the following simpler claim: Given an abelian variety A and a quasi-isogeny of p-divisible groups $f : A[p^\infty] \to X$, there is a unique triple (B, α, g) where B is an abelian variety, where $\alpha : B[p^\infty] \to X$ is an isomorphism and $g : A \dashrightarrow B$ is a p-power quasi-isogeny such that $\alpha \circ g = f$. The proof of this simpler claim is explained in [46, Section 6.13]. □

Lemma 4.1.3. *The dotted arrow in* (4.1.1) *exists.*

Proof. This is an immediate consequence of Lemma 4.1.2 and the universal property of the fiber product. □

Proposition 4.1.4. *The morphism* $\iota : \mathrm{Sh}_{G,\emptyset,U^p} \to \widehat{\mathrm{Sh}}_{G,\emptyset,U^p}$ *induced by the universal property of* $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}$ *is a closed immersion.*

We start by recalling a lemma.

Lemma 4.1.5. *If* $f : X \to Y$ *is a perfectly proper morphism between pfp algebraic spaces over* $\overline{\mathbb{F}}_p$ *that induces a bijection on* $\overline{\mathbb{F}}_p$*-points, then* f *is an isomorphism.*

Proof. This follows from [2, Corollary 6.10] and its proof. □

Proof of Proposition 4.1.4. The map ι is a morphism of perfect algebraic spaces that are perfectly proper over Sh_{G,K,U^p}, and ι is therefore perfectly proper; in particular, the image of ι is closed.

[13]To be precise, the stack $\mathrm{Sht}_{G_V,\mathcal{P}',\{\mu\}}$ is a stack of $\mathcal{L}'$-chains of polarised Dieudonné modules. By [34, Theorem 1.2], for a perfect ring R there is an equivalence of categories between $\mathcal{L}'$-chains of polarised Dieudonné modules over $W(R)$ and $\mathcal{L}'$-chains of polarised p-divisible groups over $\operatorname{Spec} R$, which gives the desired description of $\mathrm{Sht}_{G_V,\mathcal{P}',\{\mu\}}(R)$.

Let $Z \subset \widehat{\mathrm{Sh}}_{G,\emptyset,U^p}$ be the image of ι, equipped with the reduced induced scheme structure, then $\iota : \mathrm{Sh}_{G,\emptyset,U^p} \to \widehat{\mathrm{Sh}}_{G,\emptyset,U^p}$ factors through Z because $\mathrm{Sh}_{G,\emptyset,U^p}$ is reduced. We want to show that $\iota : \mathrm{Sh}_{G,\emptyset,U^p} \to Z$ is an isomorphism, and by Lemma 4.1.5 it is enough to show that it induces a bijection on $\overline{\mathbb{F}}_p$-points. Since ι is surjective, it suffices to prove injectivity.

Now [55, Corollary 6.3] tells us that a point $x \in \mathrm{Sh}_{G,\emptyset,U^p}(\overline{\mathbb{F}}_p)$ is determined by its image in $\mathrm{Sh}_{G_V,\mathcal{P}',M^p}(\overline{\mathbb{F}}_p)$ and the tensors in the Dieudonné module of its p-divisible group. The tensors are determined by the image of x in $\mathrm{Sht}_{G,\emptyset,\{\mu\}}(\overline{\mathbb{F}}_p)$. By Lemma 4.1.3, the morphism $\mathrm{Sh}_{G,\emptyset,U^p} \to \mathrm{Sh}_{G_V,\mathcal{P}',M^p}$ factors through $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}$ and so the image of x in $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\overline{\mathbb{F}}_p)$ remembers both the image of x in $\mathrm{Sh}_{G_V,\mathcal{P}',M^p}(\overline{\mathbb{F}}_p)$ and the image of x in $\mathrm{Sht}_{G,\emptyset,\{\mu\}}(\overline{\mathbb{F}}_p)$; the lemma is proved. □

Lemma 4.1.6. *The morphism $f : \widehat{\mathrm{Sh}}_{G,\emptyset,U^p} \to \mathrm{Sh}_{G_V,\mathcal{P}',M^p}$ constructed in Lemma 4.1.3 is finite.*

Proof. By the proof of Proposition 4.1.4 there is a commutative diagram

$$\begin{array}{ccc} \widehat{\mathrm{Sh}}_{G,\emptyset,U^p} & \xrightarrow{f} & \mathrm{Sh}_{G_V,\mathcal{P}',M^p} \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \chi} \\ \mathrm{Sh}_{G,K,U^p} & \xrightarrow{f'} & \mathrm{Sh}_{G_V,\mathcal{P},M^p} \end{array} \tag{4.1.2}$$

with f' finite. It suffices to show that f is quasi-finite, since its source and target are perfectly proper over $\mathrm{Sh}_{G_V,\mathcal{P},M^p}$. We will show that for $x \in \mathrm{Sh}_{G,K,U^p}(\overline{\mathbb{F}}_p)$ with image $y = f'(x)$ the map

$$f : \xi^{-1}(x) \to \chi^{-1}(y)$$

is injective, which implies the quasi-finiteness.

To prove this injectivity on fibers we return to the commutative cube from Section 4.1.1, see equation 4.1.1. The square involving the four objects with subscript G is Cartesian by construction, and the square involving the four objects with subscript G_V is Cartesian by Lemma 4.1.2. Since Cartesian squares induce isomorphisms on fibers of maps, the injectivity of the map on fibers in (4.1.2) can instead be proved for the square

$$\begin{array}{ccc} \mathrm{Sht}_{G,\emptyset,\{\mu\}} & \longrightarrow & \mathrm{Sht}_{G_V,\mathcal{P}',\{\mu\}} \\ \downarrow & & \downarrow \\ \mathrm{Sht}_{G,K,\{\mu\}} & \longrightarrow & \mathrm{Sht}_{G_V,\mathcal{P},\{\mu\}}\,. \end{array}$$

Moreover, since the spaces of shtukas of type $\{\mu\}$ sit inside the spaces of all shtukas, we can reduce to showing the injectivity of the map on fibers for

$$\begin{array}{ccc} \mathrm{Sht}_{G,\emptyset} & \longrightarrow & \mathrm{Sht}_{G_V,\mathcal{P}'} \\ \downarrow & & \downarrow \\ \mathrm{Sht}_{G,K} & \longrightarrow & \mathrm{Sht}_{G_V,\mathcal{P}}\,. \end{array}$$

Recall from the proof of Corollary 2.2.11 the Cartesian diagrams (equation (2.2.2))

$$\begin{array}{ccc} \mathrm{Sht}_{G,\emptyset} & \longrightarrow & \mathrm{Sht}_{G,K} \\ \downarrow & & \downarrow \\ \mathbf{B}L^+\mathcal{G}_\emptyset & \longrightarrow & \mathbf{B}L^+\mathcal{G}_K, \end{array} \qquad \begin{array}{ccc} \mathrm{Sht}_{G_V,\mathcal{P}'} & \longrightarrow & \mathrm{Sht}_{G_V,\mathcal{P}} \\ \downarrow & & \downarrow \\ \mathbf{B}L^+\mathcal{P}' & \longrightarrow & \mathbf{B}L^+\mathcal{P}, \end{array}$$

that fit into a commutative cube that we will not draw. This reduces the problem to showing the injectivity statement for the map on fibers in the diagram

$$\begin{array}{ccc} \mathbf{B}L^+\mathcal{G}_\emptyset & \longrightarrow & \mathbf{B}L^+\mathcal{P}' \\ \downarrow & & \downarrow \\ \mathbf{B}L^+\mathcal{G}_K & \longrightarrow & \mathbf{B}L^+\mathcal{P}, \end{array}$$

which comes down to showing injectivity of the map of partial flag varieties

$$\frac{L^+\mathcal{G}_K}{L^+\mathcal{G}_\emptyset} \to \frac{L^+\mathcal{P}}{L^+\mathcal{P}'}.$$

This last statement follows from the fact that the intersection of $L^+\mathcal{P}'$ with $L^+\mathcal{G}_K$ is equal to $L^+\mathcal{G}_\emptyset$. This is true by construction of $\mathcal{P}, \mathcal{P}'$ and the fact that $\mathcal{G}_K$ and $\mathcal{G}_\emptyset$ are connected parahoric subgroups (the first by assumption, the second by Lemma 2.2.4). □

4.2. **A perfect local model diagram.** Consider the composition (the last arrow comes from the diagram in Remark 2.3.2)

$$\widehat{\mathrm{Sh}}_{G,\emptyset,U^p} \to \mathrm{Sht}_{G,\emptyset,\{\mu\}} \to \mathrm{Sht}^{(m,1)}_{G,\emptyset,\{\mu\}} \to \left[\mathrm{M}^{\mathrm{loc}}_{\emptyset,\{\mu\}}/\overline{\mathcal{G}}_\emptyset\right].$$

We will think of this as a local model diagram for $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}$.

Proposition 4.2.1. *The morphism* $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p} \to \left[\mathrm{M}^{\mathrm{loc}}_{\emptyset,\{\mu\}}/\overline{\mathcal{G}}_\emptyset\right]$ *is weakly perfectly smooth and* $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}$ *is equidimensional of the same dimension as* $\mathrm{Sh}_{G,\emptyset,U^p}$.

Proof. We will use the results of Section 2.3. Fix $n \geq 2$ and choose $m \gg 0$ such that the action $\mathrm{Ad}_\sigma L^+\mathcal{G}_K$ on $\mathrm{M}^{\mathrm{loc,n}}_{K,\{\mu\}}$ factors through $L^m\mathcal{G}_K$ and such that m satisfies the assumption of Proposition 2.3.4. As explained in Section 3.1.7 the natural morphism

$$\mathrm{Sh}_{G,K,U^p} \to \mathrm{Sht}^{(m,n),\mathrm{loc}}_{G,K,\{\mu\}}$$

is perfectly smooth. Combining this with the discussion in Section 2.3.10, we find that (after possibly increasing n) the composition with the natural map

$$\mathrm{Sht}^{(m,n),\mathrm{loc}}_{G,K,\{\mu\}} \to \left[\frac{\ker\gamma\backslash\,\mathrm{M}^{\mathrm{loc},\infty}_{K,\{\mu\}}}{\mathrm{Ad}_\sigma\, L^m\mathcal{G}_K}\right]$$

is weakly perfectly smooth. Proposition 2.3.4 implies that the right square in the following diagram is Cartesian

$$\begin{array}{ccccc} \widehat{\mathrm{Sh}}_{G,\emptyset,U^p} & \longrightarrow & \mathrm{Sht}_{G,\emptyset,\{\mu\}} & \longrightarrow & \left[\frac{\ker\gamma\backslash\,\mathrm{M}^{\mathrm{loc},\infty}_{\emptyset,\{\mu\}}}{\mathrm{Ad}_\sigma\, H_m}\right] \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Sh}_{G,K,U^p} & \longrightarrow & \mathrm{Sht}_{G,K,\{\mu\}} & \longrightarrow & \left[\frac{\ker\gamma\backslash\,\mathrm{M}^{\mathrm{loc},\infty}_{K,\{\mu\}}}{\mathrm{Ad}_\sigma\, L^m\mathcal{G}_K}\right]. \end{array} \tag{4.2.1}$$

Since the left square is Cartesian by construction, it follows that the outer square is also Cartesian. Moreover, Lemma 2.3.6 tells us that the stack in the bottom right corner of (4.2.1) is equidimensional.

We know that Sh_{G,K,U^p} is also equidimensional and that the map

$$\mathrm{Sh}_{G,K,U^p} \to \left[\frac{\ker\gamma\backslash \mathrm{M}^{\mathrm{loc},\infty}_{K,\{\mu\}}}{\mathrm{Ad}_\sigma\, L^m\mathcal{G}_K}\right]$$

is weakly perfectly smooth. Thus by Lemma 2.1.16 this map must be weakly perfectly smooth of constant relative dimension M. Because the diagram in (4.2.1) is Cartesian, it follows that the natural map

$$\widehat{\mathrm{Sh}}_{G,\emptyset,U^p} \to \left[\frac{\ker\gamma\backslash \mathrm{M}^{\mathrm{loc},\infty}_{\emptyset,\{\mu\}}}{\mathrm{Ad}_\sigma\, H_m}\right]$$

is also weakly perfectly smooth of constant relative dimension M. By Lemma 2.3.6, both stacks in the rightmost column of (4.2.1) are equidimensional of the same dimension. We deduce from Lemma 2.1.16 that $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}$ is equidimensional of the same dimension as Sh_{G,K,U^p} and thus equidimensional of the same dimension as $\mathrm{Sh}_{G,\emptyset,U^p}$.

After possibly increasing m, we may choose $0 \ll m' \ll m$ and invoke Lemma 2.3.9, which tells us that the natural map

$$\left[\frac{\ker\gamma\backslash \mathrm{M}^{\mathrm{loc},\infty}_{\emptyset,\{\mu\}}}{\mathrm{Ad}_\sigma\, H_m}\right] \to \mathrm{Sht}^{(m',1)}_{G,\emptyset,\{\mu\}}$$

is weakly perfectly smooth. It follows from [50, Proposition 4.2.5] that the natural map

$$\mathrm{Sht}^{(m',1)}_{G,\emptyset,\{\mu\}} \to \left[\mathrm{M}^{\mathrm{loc}}_{\emptyset,\{\mu\}}/\overline{\mathcal{G}}_\emptyset\right]$$

is weakly perfectly smooth. Therefore the map $\hat\lambda : \widehat{\mathrm{Sh}}_{G,\emptyset,U^p} \to \left[\mathrm{M}^{\mathrm{loc}}_{\emptyset,\{\mu\}}/\overline{\mathcal{G}}_\emptyset\right]$ is a composition of weakly perfectly smooth maps, and hence weakly perfectly smooth. □

4.2.2. For $w \in \mathrm{Adm}(\{\mu\})$, we define the Kottwitz–Rapoport (KR) stratum $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(w)$ to be the inverse image of the locally closed substack

$$\left[\mathrm{M}^{\mathrm{loc}}_{\emptyset,\{\mu\}}(w)/\overline{\mathcal{G}}_\emptyset\right] \subset \left[\mathrm{M}^{\mathrm{loc}}_{\emptyset,\{\mu\}}/\overline{\mathcal{G}}_\emptyset\right]$$

under the weakly perfectly smooth map $\hat\lambda : \widehat{\mathrm{Sh}}_{G,\emptyset,U^p} \to \left[\mathrm{M}^{\mathrm{loc}}_{\emptyset,\{\mu\}}/\overline{\mathcal{G}}_\emptyset\right]$. Similarly, we define

$$\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\leq w) := \bigcup_{w'\leq w} \widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(w'), \tag{4.2.2}$$

which is the same as the closure of $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(w)$ because $\hat\lambda$ is open and since the closure relations hold on $\mathrm{M}^{\mathrm{loc}}_{\emptyset,\{\mu\}}$, see Section 2.2.12.

Corollary 4.2.3. *For $w \in \mathrm{Adm}(\{\mu\})$, the closure $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\leq w)$ has dimension $\ell(w)$ and is locally integral.*

Proof. Let $d = \mathrm{Dim}\,\mathrm{Sh}_{G,\emptyset,U^p} = \mathrm{Dim}\,\mathrm{M}^{\mathrm{loc}}_{\emptyset,\{\mu\}}$. Then the local model $\mathrm{M}^{\mathrm{loc}}_{\emptyset,\{\mu\}}$ is the union of $\mathrm{M}^{\mathrm{loc}}_{\emptyset,\{\mu\}}(\leq w)$ for $w \in \mathrm{Adm}(\{\mu\})$ of length d, and for such w the KR stratum $\mathrm{M}^{\mathrm{loc}}_{\emptyset,\{\mu\}}(\leq w)$ is equidimensional of

dimension d and stable under the action of $\overline{\mathcal{G}}_\emptyset$. Using $\hat{\lambda}$, we see that

$$\widehat{\mathrm{Sh}}_{G,\emptyset,U^p} = \bigcup_{\substack{w\in \mathrm{Adm}(\{\mu\})\\ \ell(w)=d}} \widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\leq w),$$

and since $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}$ is equidimensional of dimension d, it follows that for w with $\ell(w)=d$ we have that $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\leq w)$ is equidimensional of dimension $d=\ell(w)$. We can now apply Lemma 2.1.16 to deduce that $\hat{\lambda}$ is weakly perfectly smooth of relative dimension 0. We can apply Lemma 2.1.16 again to deduce the dimension results for $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\leq w)$ for arbitrary w, from the dimension results for $\mathrm{M}^{\mathrm{loc}}_{\emptyset,\{\mu\}}(\leq w)$ from Section 2.2.12.

The morphism $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p} \to \left[\mathrm{M}^{\mathrm{loc}}_{\emptyset,\{\mu\}}/\overline{\mathcal{G}}_\emptyset\right]$ is (by definition) the same as a diagram

$$\begin{array}{ccc} & \widetilde{\widehat{\mathrm{Sh}}}_{G,\emptyset,U^p} & \\ {\scriptstyle s}\swarrow & & \searrow{\scriptstyle t} \\ \widehat{\mathrm{Sh}}_{G,\emptyset,U^p} & & \mathrm{M}^{\mathrm{loc}}_{\emptyset,\{\mu\}}, \end{array}$$

where $s:\widetilde{\widehat{\mathrm{Sh}}}_{G,\emptyset,U^p}\to\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}$ is a $\overline{\mathcal{G}_\emptyset}=L^1\mathcal{G}_\emptyset$-torsor. To show that the local rings of $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\leq w)$ are integral, it suffices to check this after the perfectly smooth (in particular fpqc) cover s. Since $t:\widetilde{\widehat{\mathrm{Sh}}}_{G,\emptyset,U^p}(\leq w)\to \mathrm{M}^{\mathrm{loc}}_{\emptyset,\{\mu\}}(\leq w)$ is weakly perfectly smooth, the result follows from Lemma 2.1.10. □

We now give a corollary of Lemma 4.1.6

Corollary 4.2.4. *For $w\in\mathrm{Adm}(\{\mu\})$, the KR stratum $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(w)$ is quasi-affine.*

Proof. Section 3.1.10 and in particular equation (3.1.3) shows that there is a commutative diagram where all the horizontal maps are finite

$$\begin{array}{ccccc} \mathscr{S}_{U'}(G,X) & \longrightarrow & \mathscr{S}_{M^pM'_p}(G_V,\mathcal{H}_V)\otimes\mathcal{O}_{E,(v)} & \longrightarrow & \mathscr{S}_{J'^pJ'_p}(G_{V_{\mathcal{L}'}},\mathcal{H}_{V_{\mathcal{L}'}})\otimes\mathcal{O}_{E,(v)} \\ \downarrow{\scriptstyle \pi_{\emptyset,K}} & & \downarrow & & \\ \mathscr{S}_{U}(G,X) & \longrightarrow & \mathscr{S}_{M^pM_p}(G_V,\mathcal{H}_V)\otimes\mathcal{O}_{E,(v)} & \longrightarrow & \mathscr{S}_{J^pJ_p}(G_{V_{\mathcal{L}}},\mathcal{H}_{V_{\mathcal{L}}})\otimes\mathcal{O}_{E,(v)}. \end{array}$$

Using Zarhin's trick as in [50, Remark 2.1.4] or [27, Section 1.3.3]), there is moreover a finite map $\mathscr{S}_{J'^pJ'_p}(G_{V_{\mathcal{L}'}}\mathcal{H}_{V_{\mathcal{L}'}})\to\mathscr{S}_{Q^pQ_p}(G_{V''},\mathcal{H}_{V''})$, where $V''=V_{\mathcal{L}'}^{\oplus 4}\oplus V_{\mathcal{L}'}^{*,\oplus 4}$ and where ψ'' is given by a certain explicit matrix. Here Q_p corresponds to the self dual lattice $V_{\mathcal{L}',p}^{\oplus 4}\oplus V_{\mathcal{L}',p}^{*,\oplus 4}$ and $Q^p\subset G_{V''}(\mathbb{A}_f^p)$ is sufficiently small. By Lemma 4.1.6, the pullback $\mathcal{E}$ of the (ample) Hodge bundle from the perfection of $\mathscr{S}_{Q^pQ_p,\overline{\mathbb{F}}_p}(G_{V''},\mathcal{H}_{V''})$ to $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}$ is ample.

By construction, see Lemma 4.1.2, the left square in the following diagram commutes

$$\begin{array}{ccccc} \widehat{\mathrm{Sh}}_{G,\emptyset,U^p} & \longrightarrow & \mathrm{Sh}_{G_V,\mathcal{P}',M^p} & \longrightarrow & \mathscr{S}_{Q^pQ_p,\overline{\mathbb{F}}_p}(G_{V''},\mathcal{H}_{V''})^{\mathrm{perf}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Sht}_{G,\emptyset,\{\mu\}} & \longrightarrow & \mathrm{Sht}_{G_V,\mathcal{P}',\{\mu\}} & \longrightarrow & \mathrm{Sht}_{G_{V''},Q_p,\{\mu\}}. \end{array}$$

The right square moreover commutes because Zarhin's trick is given by a morphism of Shimura data, and then we can use [43, Corollary 4.3.2].

The arguments in the proof of [50, Theorem 3.5.9] now show that the restriction of $\mathcal{E}$ to the KR stratum $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(w)$ for $w \in \mathrm{Adm}(\{\mu\})$ is torsion, from which it follows that $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(w)$ is quasi-affine. To elaborate, their arguments show that the Hodge bundle on $\mathscr{S}_{Q^pQ_p,\overline{\mathbb{F}}_p}(G_{V''},\mathcal{H}_{V''})^{\mathrm{perf}}$ comes via pullback from a line bundle $\mathcal{F}$ on $\mathrm{Sht}_{G_{V''},Q_p,\{\mu\}}$. They then show that if we pull back $\mathcal{F}$ to $\mathrm{Sht}_{G,\emptyset,\{\mu\}}$ and restrict to a KR stratum, that the resulting line bundle is torsion. The same therefore holds for the restriction of $\mathcal{E}$ to $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(w)$, which implies that $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(w)$ is quasi-affine. □

4.3. **Connected components of closures of KR strata.** The goal of this section is to understand, for $w \in \mathrm{Adm}(\{\mu\})$, the fibers of

$$\pi_0(\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\leq w)) \to \pi_0(\mathrm{Sh}_{G,K,U^p}).$$

Here $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\leq w)$ is the closure of the KR stratum $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(w)$, see equation (4.2.2). We will eventually reduce this to understanding the fibers of

$$\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\tau) \to \pi_0(\mathrm{Sh}_{G,K,U^p}),$$

where $\tau \in \mathrm{Adm}(\{\mu\})$ is the unique element of length zero. To make this reduction, we will show that each connected component of $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\leq w)$ intersects $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\tau)$. This will require us to assume that either $\mathbf{Sh}_U(G,X)$ is proper or that $G_{\mathbb{Q}_p}$ is unramified. More generally, we require that Conjecture 4.3.1 below holds. Recall that there are EKOR strata $\mathrm{Sh}_{G,K,U^p}\{w\}$ for $w \in {}^K\mathrm{Adm}(\{\mu\})$, see Section 2.3.11 and Section 3.1.7, with closures $\mathrm{Sh}_{G,K,U^p}\{\preceq w\}$.

Conjecture 4.3.1. *If V is an irreducible component $\mathrm{Sh}_{G,K,U^p}\{\preceq w\}$ for some $w \in {}^K\mathrm{Adm}(\{\mu\})$, then V intersects the unique 0-dimensional EKOR stratum $\mathrm{Sh}_{G,K,U^p}\{\tau\}$.*

Remark 4.3.2. When $\mathcal{G}_K$ is hyperspecial, then this Conjecture is [53, Proposition 6.20]; the assumption made in the statement of this proposition is proved in [1]. When $\mathbf{Sh}_U(G,X)$ is proper, we will circumvent the conjecture using Lemma 4.3.4 below. This is where the 'either unramified or proper' assumption in Theorems 2, 3 and 4 comes from.

4.3.3. We start by proving a lemma.

Lemma 4.3.4. *Let Z be a connected component of $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\leq w)$. Suppose that there exists a KR stratum $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(x)$ such that $Z \cap \widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(x)$ is non-empty and such that $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\leq x)$ is perfectly proper over $\operatorname{Spec} k$. Then Z intersects $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\tau)$.*

Proof. Let $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(x)$ be as in the statement of the lemma. Then there is an $x' \leq x$ of minimal length such that $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(x') \cap Z \neq \emptyset$, and it suffices to prove that this length is equal to zero. The minimality tells us that

$$\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(x') \cap Z = \widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\leq x') \cap Z, \tag{4.3.1}$$

since $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\leq x') \setminus \widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(x')$ is a union of KR strata associated to $x'' \in \mathrm{Adm}(\{\mu\})$ of length strictly smaller than x'. Next, we note that $Z \cap \widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(x')$ is a union of connected components of $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(x')$, because $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(x') \subset \widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\leq w)$ and so connected components of $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(x')$ are either disjoint from Z or contained in Z.

Since $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(x')$ is quasi-affine by Corollary 4.2.4, we find that $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(x') \cap Z$ is quasi-affine. Moreover (4.3.1) implies that $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(x') \cap Z \subset \widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\leq x)$ is closed, hence perfectly proper over $\operatorname{Spec} k$. Therefore $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(x') \cap Z$ is perfectly proper and quasi-affine, and thus zero-dimensional. Since it is a union of connected components of $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(x')$, it follows from Corollary 4.2.3 that x' has length zero and must therefore be equal to τ. □

We will deduce the same result from Conjecture 4.3.1 when the Shimura variety is not proper.

Proposition 4.3.5. *If Conjecture 4.3.1 holds, then for $w \in \mathrm{Adm}(\{\mu\})$ every connected component Z of $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\leq w)$ intersects $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\tau)$.*

First, we prove two lemmas.

Lemma 4.3.6. *Let $Z \subset \widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\leq w)$ be a connected component. If $x \in \mathrm{Adm}(\{\mu\})$ is of minimal length with the property that $Z \cap \widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(x) \neq \emptyset$, then x is σ-straight in the sense of* [19, Section 1.3].

Proof. Arguing as in the proof of Lemma 4.3.4 above, we see that the intersection $Z \cap \widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(x)$ is a union of connected components of $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(x)$. Let V be one of these components, then V is closed in $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\leq x)$ as in the proof of Lemma 4.3.4. Moreover, V is actually a connected component of $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\leq x)$; it is an irreducible component for dimension reasons and thus a connected component since $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\leq x)$ is locally integral (see Corollary 4.2.3).

Let $z \in V(\overline{\mathbb{F}}_p)$ with image $\pi(z) \in \mathrm{Sh}_{G,K,U^p}(\overline{\mathbb{F}}_p)$, and consider the uniformisation map

$$i_{\pi(z)} : X(\mu, b)_K(\overline{\mathbb{F}}_p) \to \mathrm{Sh}_{G,K,U^p}(\overline{\mathbb{F}}_p),$$

centered at $\pi(z)$, where b corresponds to $\pi(z)$. By the proof of [20, Proposition 5.2.2] we can upgrade this to a morphism of perfect schemes $i_{\pi(z)} : X(\mu, b)_K \to \mathrm{Sh}_{G,K,U^p}$. As in the proof of Theorem 3.3.1, see the discussion in Section 4.3.8 below, it follows that there is an induced map

$$i_z : X(\mu, b)_\emptyset \to \widehat{\mathrm{Sh}}_{G,\emptyset,U^p}$$

whose image contains z. Since the uniformisation map is compatible with the KR stratification, this restricts to a map

$$X(\mu, b)_\emptyset(\leq x) \to \widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\leq x)$$

whose image contains z. This means that there is a connected component Y of $X(\mu, b)_\emptyset(\leq x)$ that maps to V'. Now [19, Theorem 4.1] tells us that there is a σ-straight element $x' \leq x$ in $\mathrm{Adm}(\{\mu\})$ such that $Y \cap X(\mu, b)_\emptyset(x') \neq \emptyset$. In particular, $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(x') \cap V \neq \emptyset$ and so $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(x') \cap Z \neq \emptyset$. Since x has been chosen to be minimal with the property that $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(x') \cap Z \neq \emptyset$, we see that $x = x'$ and so x is σ-straight. □

Lemma 4.3.7. *Let $x \in \mathrm{Adm}(\{\mu\})$ be σ-straight. Then there is $y \in {}^K\mathrm{Adm}(\{\mu\})$ such that the natural map $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(w) \to \mathrm{Sh}_{G,K,U^p}$ factors via a finite étale map $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(w) \to \mathrm{Sh}_{G,K,U^p}\{y\}$ and such that $\ell(y) = \ell(w)$.*

Proof. By the proof of [16, Theorem 6.17], there is an element $v \in W_K$ such that $y := vx\sigma(v)^{-1}$ lies in ${}^K\mathrm{Adm}(\{\mu\})$ and such that $\ell(y) = \ell(x)$. It follows from [16, the discussion prior to Theorem 6.10] that the image of $\mathrm{Sht}_{G,\emptyset,\{\mu\}}(x)(\overline{\mathbb{F}}_p)$ in $\mathrm{Sht}_{G,K}(\overline{\mathbb{F}}_p)$ is equal to $\mathrm{Sht}_{G,K,\{\mu\}}\{y\}(\overline{\mathbb{F}}_p)$.[14] Since KR strata and EKOR strata on $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}$ and Sh_{G,K,U^p} respectively are defined as the inverse images of KR strata and EKOR strata in $\mathrm{Sht}_{G,\emptyset,\{\mu\}}$ and $\mathrm{Sht}_{G,K,\{\mu\}}$, and because these strata are determined by their $\overline{\mathbb{F}}_p$-points, we deduce that the image of $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(x) \to \mathrm{Sh}_{G,K,U^p}$ is equal to $\mathrm{Sh}_{G,K,U^p}\{y\}$.

To prove that the induced map is finite étale, we may use diagram (4.0.1) to reduce to checking finite étale-ness of $\mathrm{Sht}_{G,\emptyset,\{\mu\}}(w) \to \mathrm{Sht}_{G,K,\{\mu\}}\{y\}$, and then Lemma 2.4.5 to reduce to checking this for $X(\mu, b)_\emptyset(w) \to X(\mu, b)_K\{y\}$. The latter holds because $J_b(\mathbb{Q}_p)$ acts transitively on $X(\mu, b)_\emptyset(w)(\overline{\mathbb{F}}_p)$ by [19, Theorem 5.1], with stabiliser a compact open subgroup, cf. [56, Proposition 3.1.4], and the

[14] Recall that for $y \in {}^K\mathrm{Adm}(\{\mu\})$ we use $\{y\}$ to denote the corresponding EKOR stratum, see Section 2.3.11.

same holds for $X(\mu,b)_K\{w\}(\overline{\mathbb{F}}_p)$. Thus both $X(\mu,b)_\emptyset(w)$ and $X(\mu,b)_K\{y\}$ are isomorphic to infinite disjoint unions of $\operatorname{Spec}\overline{\mathbb{F}}_p$ and the map $X(\mu,b)_K(w)\to X(\mu,b)_K\{y\}$ is quasi-finite. It is not hard to see that this implies that $X(\mu,b)_K(w)\to X(\mu,b)_K\{y\}$ is finite étale. □

Proof of Proposition 4.3.5. Let $x\in \mathrm{Adm}(\{\mu\})$ be of minimal length with the property that $Z\cap \widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(x)\neq\emptyset$, we would like to show that $\ell(x)=0$. Arguing as in the proof of Lemma 4.3.4 above, we see that the intersection $Z\cap\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(x)$ is a union of connected components of $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(x)$ and that $Z\cap\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(x)$ is closed in $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\leq x)$. Let V be one of these components, then V has dimension $\ell(x)$ and V is closed inside $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\leq x)$. Thus V must be an irreducible component of $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\leq x)$.

By Lemma 4.3.6 we see that x is σ-straight. By Lemma 4.3.7 there exists $y\in {}^K\mathrm{Adm}(\{\mu\})$ such that the natural map $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(w)\to\mathrm{Sh}_{G,K,U^p}$ factors via a finite étale map $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(w)\to\mathrm{Sh}_{G,K,U^p}\{y\}$ and such that $\ell(x)=\ell(y)$. We conclude that the image of V in $\mathrm{Sh}_{G,K,U^p}\{y\}$ is an irreducible component of $\mathrm{Sh}_{G,K,U^p}\{y\}$. Since V is closed in $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\leq x)$ and thus in $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}$, and since the map $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}\to\mathrm{Sh}_{G,K,U^p}$ is perfectly proper, it follows that $\pi(V)$ is closed in Sh_{G,K,U^p}. Therefore $\pi(V)$ is closed inside $\mathrm{Sh}_{G,K,U^p}\{\preceq y\}$, the closure of $\mathrm{Sh}_{G,K,U^p}\{y\}$, and therefore an irreducible component of $\mathrm{Sh}_{G,K,U^p}\{\preceq y\}$.

Conjecture 4.3.1 tells us that $\pi(V)$ intersects the zero-dimensional EKOR stratum $\mathrm{Sh}_{G,K,U^p}\{\tau\}$, and since $\pi(V)\subset\mathrm{Sh}_{G,K,U^p}\{y\}$ it follows that $\tau=y$ and so that $0=\ell(y)=\ell(x)$. It follows that $x=\tau$ and so we are done. □

4.3.8. We will explicitly analyse the basic KR stratum $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\tau)$, where $\tau\in\mathrm{Adm}(\{\mu\})$ is the unique element of length zero. Let $x\in\widehat{\mathrm{Sh}}_{G,\emptyset}(\tau)(\overline{\mathbb{F}}_p)$ with image $\pi(x)\in\mathrm{Sh}_{G,K,U^p}(\overline{\mathbb{F}}_p)$ and choose an isomorphism $\mathbb{D}_x\simeq V_{\mathbb{Z}_p}\otimes_{\mathbb{Z}_p}\breve{\mathbb{Z}}_p$ sending $s_{\alpha,0,x}$ to $s_\alpha\otimes 1$. Let $b\in G(\breve{\mathbb{Q}}_p)$ be the element corresponding to the Frobenius of $\mathbb{D}_x$ under this isomorphism. Let I_x be the algebraic group $I_{\pi(x)}$ introduced in Section 3.1.9 and let

$$j_x^p: I_{x,\mathbb{A}_f^p}\to G_{\mathbb{A}_f^p}$$
$$j_{x,p}: I_{x,\mathbb{Q}_p}\to J_b$$

be the maps induced by the choices made above. Then by [20, Proposition 5.2.2], there is an isomorphism of perfect schemes (where $\mathrm{Sh}_{G,K,[b],U^p}\subset\mathrm{Sh}_{G,K,U^p}$ denotes the Newton stratum associated to $[b]$)

$$i_{\pi(x)}: I_x(\mathbb{Q})\backslash X(\mu,b)_K\times G(\mathbb{A}_f^p)/U^p\to\mathrm{Sh}_{G,K,[b],U^p},$$

where $I_x(\mathbb{Q})$ acts on $G(\mathbb{A}_f^p)$ via j_x^p and on $X(\mu,b)_K$ via $j_{x,p}: I(\mathbb{Q})\to J_b(\mathbb{Q}_p)$ and then the natural action of $J_b(\mathbb{Q}_p)$ on $X(\mu,b)_K$. Moreover, it follows from [20, Proposition 5.2.6] that j_x^p and $j_{x,p}$ are isomorphisms and that $I(\mathbb{R})$ is compact mod centre.

Consider the Cartesian diagram

$$\begin{array}{ccc}\widehat{\mathrm{Sh}}_{G,\emptyset,[b],U^p} & \longrightarrow & \mathrm{Sht}_{G,\emptyset,\mu,[b]}\\ \downarrow & & \downarrow\\ \mathrm{Sh}_{G,K,[b],U^p} & \longrightarrow & \mathrm{Sht}_{G,K,\{\mu\},[b]},\end{array}\tag{4.3.2}$$

Applying Lemma 2.4.5 to $\mathrm{Sht}_{G,\emptyset,\mu,[b]}$ and $\mathrm{Sht}_{G,K,\{\mu\},[b]}$ and using i_x we can identify (4.3.2) with

$$\begin{array}{ccc} \widehat{\mathrm{Sh}}_{G,\emptyset,[b],U^p} & \longrightarrow & \left[\frac{X(\mu,b)_\emptyset}{\underline{J_b(\mathbb{Q}_p)}}\right] \\ \downarrow & & \downarrow \\ I_x(\mathbb{Q})\backslash X(\mu,b)_K \times G(\mathbb{A}_f^p)/U^p & \longrightarrow & \left[\frac{X(\mu,b)_K}{\underline{J_b(\mathbb{Q}_p)}}\right]. \end{array}$$

By Lemma 3.3.3 the map (induced by the bottom horizontal map)

$$X(\mu,b)_K \times G(\mathbb{A}_f^p) \to \left[\frac{X(\mu,b)_K}{\underline{J_b(\mathbb{Q}_p)}}\right]$$

is the natural projection map onto the first factor followed by the natural map to the quotient. As in the proof of Theorem 3.3.1, it follows that there is an isomorphism

$$i_x : I_x(\mathbb{Q})\backslash X(\mu,b)_\emptyset \times G(\mathbb{A}_f^p)/U^p \to \widehat{\mathrm{Sh}}_{G,\emptyset,[b],U^p}$$

such that the map (coming from the left vertical map in (4.3.2))

$$I_x(\mathbb{Q})\backslash X(\mu,b)_\emptyset \times G(\mathbb{A}_f^p)/U^p \to I_x(\mathbb{Q})\backslash X(\mu,b)_K \times G(\mathbb{A}_f^p)/U^p,$$

is induced by the natural projection $X(\mu,b)_\emptyset \to X(\mu,b)_K$ and the identity of $G(\mathbb{A}_f^p)$.

4.3.9. To analyse the fibers of $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\tau) \to \pi_0(\mathrm{Sh}_{G,K,U^p})$, we will first analyse the fibers of $X(\mu,b)_\emptyset(\tau) \to \pi_1(G)_I^\sigma$. Let $J_b^{\mathrm{sc}} \to J_b^{\mathrm{der}}$ be the simply connected cover of the derived group of J_b.

Lemma 4.3.10. *The group $J_b^{\mathrm{sc}}(\mathbb{Q}_p)$ acts transitively on the fibers of*

$$X(\mu,b)_\emptyset(\tau) \to \pi_1(G)_I^\sigma.$$

Proof. The element τ is σ-straight and so $J_b(\mathbb{Q}_p)$ acts transitively on $X(\mu,b)_\emptyset(\tau)$ by $J_b(\mathbb{Q}_p)$ by [17, Theorem 4.8]. The stabiliser of a point is a parahoric subgroup $N_p \subset J_b(\mathbb{Q}_p)$ by [56, Proposition 3.1.4]. Therefore our map can be identified with the natural map

$$X(\mu,b)_\emptyset(\tau) = \frac{J_b(\mathbb{Q}_p)}{N_p} \to \frac{J_b(\mathbb{Q}_p)}{N_p J^{\mathrm{sc}}(\mathbb{Q}_p)} = \pi_1(G)_I^\sigma,$$

using Lemma 3.4.2 and the fact that b is basic in the last step, and the result follows. □

4.3.11. The goal of this subsection is to prove an auxiliary result. Let $\mathbb{G}$ and $\mathbb{H}$ be connected reductive groups over $\mathbb{Q}$ that are inner forms of each other, and such that they are isomorphic over $\mathbb{A}_f^p$. Fix an identification $\mathbb{G} \otimes \mathbb{A}_f^p \simeq \mathbb{H} \otimes \mathbb{A}_f^p$ and an inner twisting $\Psi : \mathbb{G}_{\overline{\mathbb{Q}}_p} \to \mathbb{H}_{\overline{\mathbb{Q}}_p}$, which induces an isomorphism $\pi_1(\mathbb{G})_I^\sigma \simeq \pi_1(\mathbb{H})_I^\sigma$. Recall the notation $\mathbb{G}(\mathbb{R})_+$ and $\mathbb{G}(\mathbb{Q})_+$ from Section 3.4.

Proposition 4.3.12 (Borovoi)**.** *The images of $\mathbb{G}(\mathbb{Q})_+$ and $\mathbb{H}(\mathbb{Q})_+$ in*

$$\frac{\mathbb{G}(\mathbb{A}_f^p)}{\rho(\mathbb{G}^{\mathrm{sc}}(\mathbb{A}_f^p))} \times \pi_1(\mathbb{G})_I^\sigma$$

are equal (after applying our fixed identifications).

The following arguments have been reproduced and adapted with permission from Mikhail Borovoi's Mathoverflow answer [4]; we will use [3, Section 3]. We consider the *crossed module* $(\mathbb{G}^{\mathrm{sc}} \to \mathbb{G})$ and the *hypercohomology*

$$H^0_{\mathrm{ab}}(\mathbb{Q},\mathbb{G}) := H^0(\mathbb{Q},\mathbb{G}^{\mathrm{sc}} \to \mathbb{G}),$$

where $\mathbb{G}$ is in degree 0; see [3]. The cohomology set $H^0_{\mathrm{ab}}(\mathbb{Q},\mathbb{G})$ is naturally an abelian group that does not change under inner twisting of $\mathbb{G}$. The short exact sequence

$$1 \to (1 \to \mathbb{G}) \to (\mathbb{G}^{\mathrm{sc}} \to \mathbb{G}) \to (\mathbb{G}^{\mathrm{sc}} \to 1) \to 1$$

induces a hypercohomology exact sequence

$$\mathbb{G}^{\mathrm{sc}}(\mathbb{Q}) \to \mathbb{G}(\mathbb{Q}) \to H^0_{\mathrm{ab}}(\mathbb{Q},\mathbb{G}) \to H^1(\mathbb{Q},\mathbb{G}^{\mathrm{sc}}),$$

where

$$\mathrm{ab}^0 \colon \mathbb{G}(\mathbb{Q}) \to H^0_{\mathrm{ab}}(\mathbb{Q},\mathbb{G})$$

is the *abelianisation map*. Let Z be the center of $\mathbb{G}$, then by definition,

$$\mathbb{G}(\mathbb{R})_+ = Z(\mathbb{R}) \cdot \rho(\mathbb{G}^{\mathrm{sc}}(\mathbb{R})),$$

and hence

$$\mathbb{G}(\mathbb{R})_+/\rho(\mathbb{G}^{\mathrm{sc}}(\mathbb{R})) = \mathrm{ab}^0(Z(\mathbb{R})) \subset \ker[H^0_{\mathrm{ab}}(\mathbb{R},\mathbb{G}) \to H^1(\mathbb{R},\mathbb{G}^{\mathrm{sc}})].$$

We see that the image of $\mathbb{G}(\mathbb{Q})_+$ in $H^0_{\mathrm{ab}}(\mathbb{Q},\mathbb{G})$ can be identified with the preimage $\mathrm{ab}^0(Z(\mathbb{R})) \subset H^0_{\mathrm{ab}}(\mathbb{R},\mathbb{G})$ in $\ker[H^0_{\mathrm{ab}}(\mathbb{Q},\mathbb{G}) \to H^1(\mathbb{Q},\mathbb{G}^{\mathrm{sc}})]$ under the natural map

$$f : \ker[H^0_{\mathrm{ab}}(\mathbb{Q},\mathbb{G}) \to H^1(\mathbb{Q},\mathbb{G}^{\mathrm{sc}})] \to \ker[H^0_{\mathrm{ab}}(\mathbb{R},\mathbb{G}) \to H^1(\mathbb{R},\mathbb{G}^{\mathrm{sc}})].$$

Lemma 4.3.13. *The preimage of* $\mathrm{ab}^0(Z(\mathbb{R})) \subset H^0_{\mathrm{ab}}(\mathbb{R},\mathbb{G})$ *in* $\ker[H^0_{\mathrm{ab}}(\mathbb{Q},\mathbb{G}) \to H^1(\mathbb{Q},\mathbb{G}^{\mathrm{sc}})]$ *under* f *coincides with the preimage of* $\mathrm{ab}^0(Z(\mathbb{R}))$ *in* $H^0_{\mathrm{ab}}(\mathbb{Q},\mathbb{G})$.

Proof. Let $\xi \in H^0_{\mathrm{ab}}(\mathbb{Q},\mathbb{G})$ lie in the preimage of

$$\mathrm{ab}^0(Z(\mathbb{R})) \subset \ker[H^0_{\mathrm{ab}}(\mathbb{R},\mathbb{G}) \to H^1(\mathbb{R},\mathbb{G}^{\mathrm{sc}})].$$

Then the image of ξ in $H^1(\mathbb{R},\mathbb{G}^{\mathrm{sc}})$ is trivial, and therefore, the image of ξ in $H^1(\mathbb{Q},\mathbb{G}^{\mathrm{sc}})$ lies in the kernel of the localisation map

$$H^1(\mathbb{Q},\mathbb{G}^{\mathrm{sc}}) \to H^1(\mathbb{R},\mathbb{G}^{\mathrm{sc}}).$$

By the Hasse principle for simply connected groups ([44, Theorem 6.6]), this kernel is trivial. Thus the image of ξ in $H^1(\mathbb{Q},\mathbb{G}^{\mathrm{sc}})$ is trivial, and hence ξ lies in the preimage of $\mathrm{ab}^0(Z(\mathbb{R}))$ in $\ker[H^0_{\mathrm{ab}}(\mathbb{Q},\mathbb{G}) \to H^1(\mathbb{Q},\mathbb{G}^{\mathrm{sc}})]$, as required. □

Corollary 4.3.14. *The image of the abelianisation map* $\mathbb{G}(\mathbb{Q})_+ \to H^0_{\mathrm{ab}}(\mathbb{Q},\mathbb{G})$ *is the preimage of* $\mathrm{ab}^0(Z(\mathbb{R})) \subset H^0_{\mathrm{ab}}(\mathbb{R},\mathbb{G})$ *in* $H^0_{\mathrm{ab}}(\mathbb{Q},\mathbb{G})$.

Proof of Proposition 4.3.12. It is clear from Corollary 4.3.14 and the discussion above that the image of $\mathbb{G}(\mathbb{Q})_+ \to H^0_{\mathrm{ab}}(\mathbb{Q},\mathbb{G})$ is the same for all inner forms. Thus the images of $\mathbb{H}(\mathbb{Q})_+$ and $\mathbb{G}(\mathbb{Q})_+$ in $H^0_{\mathrm{ab}}(\mathbb{Q},\mathbb{G}) = H^0_{\mathrm{ab}}(\mathbb{Q},\mathbb{H})$ are equal.

To prove the proposition, we simply note that the following diagram commutes

$$\begin{array}{ccc} \frac{\mathbb{G}(\mathbb{Q})_+}{\rho(\mathbb{G}^{\mathrm{sc}}(\mathbb{Q}))} & \hookrightarrow & H^0_{\mathrm{ab}}(\mathbb{Q},\mathbb{G}) \\ \downarrow & & \downarrow \\ \frac{\mathbb{G}(\mathbb{A}_f)}{\rho(\mathbb{G}^{\mathrm{sc}}(\mathbb{A}_f))} & \hookrightarrow & \prod_{v\neq\infty} H^0_{\mathrm{ab}}(\mathbb{Q}_v,\mathbb{G}). \end{array}$$

and that $\pi_1(\mathbb{G})^\sigma_I$ is a quotient of $\mathbb{G}(\mathbb{Q}_p)/\rho(\mathbb{G}^{\mathrm{sc}}(\mathbb{Q}_p))$ by Lemma 3.4.2. □

Proposition 4.3.15. *Let* Σ *be a finite set of primes with* $p \in \Sigma$. *Then* $G^{\mathrm{sc}}(\mathbb{A}_f^\Sigma)$ *acts transitively on the fibers of*

$$\widehat{\mathrm{Sh}}_{G,\emptyset}(\tau) \to \pi_0(\mathrm{Sh}_{G,K}).$$

Proof. Let $z \in \widehat{\mathrm{Sh}}_{G,\emptyset}(\tau)(\overline{\mathbb{F}}_p)$, where $\widehat{\mathrm{Sh}}_{G,\emptyset}(\tau) := \varprojlim_{U^p} \widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\tau)$, with image $x \in \mathrm{Sh}_{G,K}(\overline{\mathbb{F}}_p)$. Choose an isomorphism $\mathbb{D}_x \simeq V_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p$ sending $s_{\alpha,0,x}$ to $s_\alpha \otimes 1$ and let $b \in G(\breve{\mathbb{Q}}_p)$ be the element corresponding to the Frobenius of $\mathbb{D}_x$ under this isomorphism. Then as explained in Section 4.3.8, we get an isomorphism

$$i_z : \varprojlim_{U^p} I_x(\mathbb{Q})\backslash X(\mu,b)_\emptyset \times G(\mathbb{A}_f^p)/U^p \to \varprojlim_{U^p} \widehat{\mathrm{Sh}}_{G,\emptyset,[b],U^p} =: \widehat{\mathrm{Sh}}_{G,\emptyset,[b]}.$$

By Lemma 3.3.3 this induces an isomorphism

$$\varprojlim I_x(\mathbb{Q})\backslash X(\mu,b)_\emptyset(\tau) \times G(\mathbb{A}_f^p)/U^p \to \varprojlim_{U^p} \widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\tau) = \widehat{\mathrm{Sh}}_{G,\emptyset}(\tau).$$

We also note that the natural map $I_x(\mathbb{Q})\backslash X(\mu,b)_\emptyset(\tau) \times G(\mathbb{A}_f^p) \to \varprojlim_{U^p} I_x(\mathbb{Q})\backslash X(\mu,b)_\emptyset(\tau) \times G(\mathbb{A}_f^p)/U^p$ is a bijection by [37, Lemma 4.20].

Using the base point $x \in \mathrm{Sh}_{G,K}(\overline{\mathbb{F}}_p)$ to trivialise the $\pi(G)$-torsor $\pi_0(\mathrm{Sh}_{G,K})$, see the beginning of Section 3.4, we get an isomorphism

$$\begin{aligned} \pi(G) &\to \pi_0(\mathrm{Sh}_{G,K}) \\ g &\mapsto g \cdot Z_x, \end{aligned}$$

where Z_x is the connected component containing x. By the discussion in Section 3.4.1 we may identify

$$\pi(G) = G(\mathbb{Q})_+\backslash \pi_1(G)_I^\sigma \times \frac{G(\mathbb{A}_f^p)}{\rho(G^{\mathrm{sc}}(\mathbb{A}_f^p))}.$$

By Proposition 3.4.5 the map

$$\alpha_x : I_x(\mathbb{Q})\backslash X(\mu,b)_\emptyset \times G(\mathbb{A}_f^p) \to \pi_0(\mathrm{Sh}_{G,K}),$$

induced by i_x, satisfies $\alpha_x(y, g^p) = (\kappa(y), g^p) \cdot Z_x$, where $\kappa(y) \in \pi_1(G)_I^\sigma$ is the image of y and $g^p \in G(\mathbb{A}_f^p)$. Hence our identifications fit in a commutative diagram

$$\begin{array}{ccc} \widehat{\mathrm{Sh}}_{G,\emptyset}(\tau) & \xleftarrow{\sim} & I_x(\mathbb{Q})\backslash X(\mu,b)_\emptyset(\tau) \times G(\mathbb{A}_f^p) \\ \downarrow & & \downarrow \\ \pi_0(\mathrm{Sh}_{G,K}) & \xleftarrow{\sim} & G(\mathbb{Q})_+/\pi_1(G_{\mathbb{Q}_p})_I^\sigma \times \frac{G(\mathbb{A}_f^p)}{\rho(G^{\mathrm{sc}}(\mathbb{A}_f^p))}, \end{array} \tag{4.3.3}$$

where the map

$$X(\mu,b)_\emptyset(\tau) \times G(\mathbb{A}_f^p) \to \pi_1(G_{\mathbb{Q}_p})_I^\sigma \times G(\mathbb{A}_f^p)$$

is the product of the natural map κ of Section 3.4 and the identity map on $G(\mathbb{A}_f^p)$. By [44, Theorem 7.8], which is a strong approximation result, the group $I^{\mathrm{sc}}(\mathbb{Q})$ is dense in (using $j_{x,p}$ and j_x^p from Section 4.3.8 to make the identification)

$$\prod_{\ell \in \Sigma} I^{\mathrm{sc}}(\mathbb{Q}_\ell) = J_b^{\mathrm{sc}}(\mathbb{Q}_p) \prod_{\ell \in \Sigma\setminus\{p\}} G^{\mathrm{sc}}(\mathbb{Q}_\ell).$$

Recall that we sometimes write $G^{\mathrm{sc}}(\mathbb{A}_f^p) \subset G(\mathbb{A}_f^p)$ for $\rho(G^{\mathrm{sc}}(\mathbb{A}_f^p)) \subset G(\mathbb{A}_f^p)$. Using the discussion above, we can identify the right vertical map in (4.3.3) with the natural map

$$I_x(\mathbb{Q})\backslash \frac{X(\mu,b)_\emptyset(\tau)}{J_b^{\mathrm{sc}}(\mathbb{Q}_p)} \times \frac{G(\mathbb{A}_f^p)}{\prod_{\ell \in \Sigma\setminus\{p\}} G^{\mathrm{sc}}(\mathbb{Q}_\ell)} \to G(\mathbb{Q})_+/\left(\pi_1(G_{\mathbb{Q}_p})_I^\sigma \times \frac{G(\mathbb{A}_f^p)}{G^{\mathrm{sc}}(\mathbb{A}_f^p)}\right).$$

Lemma 4.3.10 tells us that κ induces an isomorphism $\frac{X(\mu,b)_\emptyset(\tau)}{J_b^{\mathrm{sc}}(\mathbb{Q}_p)} \to \pi_1(G_{\mathbb{Q}_p})_I^\sigma$ and thus we get

$$I_x(\mathbb{Q})\backslash \pi_1(G_{\mathbb{Q}_p})_I^\sigma \times \frac{G(\mathbb{A}_f^p)}{\prod_{\ell\in\Sigma\setminus\{p\}} G^{\mathrm{sc}}(\mathbb{Q}_\ell)} \to G(\mathbb{Q})_+/\left(\pi_1(G_{\mathbb{Q}_p})_I^\sigma \times \frac{G(\mathbb{A}_f^p)}{G^{\mathrm{sc}}(\mathbb{A}_f^p)}\right). \tag{4.3.4}$$

The fibers of the natural map

$$\pi_1(G_{\mathbb{Q}_p})_I^\sigma \times \frac{G(\mathbb{A}_f^p)}{\prod_{\ell\in\Sigma\setminus\{p\}} G^{\mathrm{sc}}(\mathbb{Q}_\ell)} \to \pi_1(G_{\mathbb{Q}_p})_I^\sigma \times \frac{G(\mathbb{A}_f^p)}{G^{\mathrm{sc}}(\mathbb{A}_f^p)}$$

clearly have a transitive action of $G^{\mathrm{sc}}(\mathbb{A}_f^\Sigma)$. To show that the same is true for the fibers of (4.3.4), we need to show that the images of the two natural maps

$$I_x(\mathbb{Q}), G(\mathbb{Q})_+ \to \pi_1(G_{\mathbb{Q}_p})_I^\sigma \times \frac{G(\mathbb{A}_f^p)}{G^{\mathrm{sc}}(\mathbb{A}_f^p)}. \tag{4.3.5}$$

are equal. Now note that $I_x(\mathbb{Q}) = I_x(\mathbb{Q})_+$ because $I_x^{\mathrm{ad}}(\mathbb{R})$ is compact and thus connected, see [44, Cor. 1 on page 121]. Then the required identification of the images of (4.3.5) is exactly what is proved in Proposition 4.3.12. □

Proposition 4.3.16. *Let Σ be a finite set of primes with $p \in \Sigma$. If either $\mathbf{Sh}_U(G,X)$ is proper or Conjecture 4.3.1 holds, then $G^{\mathrm{sc}}(\mathbb{A}_f^\Sigma)$ acts transitively on the fibers of*

$$\pi_0(\widehat{\mathrm{Sh}}_{G,\emptyset}(\leq w)) \to \pi_0(\mathrm{Sh}_{G,K}).$$

Proof. There is a $G(\mathbb{A}_f^p)$-equivariant commutative diagram

$$\begin{array}{ccc} \widehat{\mathrm{Sh}}_{G,\emptyset}(\tau) & \longrightarrow & \pi_0(\widehat{\mathrm{Sh}}_{G,\emptyset}(\leq w)) \\ & \searrow \quad \swarrow & \\ & \pi_0(\mathrm{Sh}_{G,K}). & \end{array} \tag{4.3.6}$$

If Conjecture 4.3.1 holds, then by Proposition 4.3.5 every connected component of $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\leq w)$ intersects $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\tau)$. If $\mathbf{Sh}_U(G,X)$ is proper, then $\mathscr{S}_U(G,X)$ is proper by the main result of [36]. Therefore Sh_{G,K,U^p} is perfectly proper and moreover $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}$ is perfectly proper since $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p} \to \mathrm{Sh}_{G,K,U^p}$ is perfectly proper. Now Lemma 4.3.4 tells us that every connected component of $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\leq w)$ intersects $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\tau)$. Thus under the assumptions of the proposition the horizontal arrow in (4.3.6) is surjective. Indeed, it is a continuous morphism of profinite sets that is a countable inverse limit of surjective maps between finite sets.

We see that the fibers of the left diagonal map surject onto the fibers of the right diagonal map. Now $G^{\mathrm{sc}}(\mathbb{A}_f^\Sigma)$ acts transitively on the fibers of the left diagonal map by Proposition 4.3.15, and therefore also on the fibers of the right diagonal map. □

4.4. **Proof of the main theorems.**

Theorem 4.4.1. *If either $\mathbf{Sh}_U(G,X)$ is proper or Conjecture 4.3.1 holds, then the natural map $\iota : \mathrm{Sh}_{G,\emptyset,U^p} \to \widehat{\mathrm{Sh}}_{G,\emptyset,U^p}$ is an isomorphism.*

Proof. We know that ι is a closed immersion by Proposition 4.1.4, whose source and target are equidimensional of the same dimension by Proposition 4.2.1. To prove that this closed immersion is an isomorphism, it suffices to show that for each $w \in \mathrm{Adm}(\{\mu\})$ of maximal length, the closed immersion

$$\mathrm{Sh}_{G,\emptyset,U^p}(\leq w) \to \widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\leq w)$$

is an isomorphism. Now source and target are locally integral by Corollary 4.2.3, and so the source is a union of connected components of the target. To show that the inclusion

$$\pi_0(\mathrm{Sh}_{G,\emptyset,U^p}(\leq w)) \to \pi_0(\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(\leq w))$$

is an isomorphism, we will use the $G(\mathbb{A}_f^p)$-equivariance of the map $\pi_0(\mathrm{Sh}_{G,\emptyset}(\leq w)) \to \pi_0(\widehat{\mathrm{Sh}}_{G,\emptyset}(\leq w))$. We know by Corollary 3.4.7 that

$$\mathrm{Sh}_{G,\emptyset,U^p}(\leq w) \to \pi_0(\mathrm{Sh}_{G,K,U^p})$$

is surjective for all U^p, and therefore it is enough to show that $G^{\mathrm{sc}}(\mathbb{A}_f^p)$ acts transitively on the fibers of $\widehat{\mathrm{Sh}}_{G,\emptyset}(\leq w) \to \pi_0(\mathrm{Sh}_{G,K})$. Under our assumptions, this follows from Proposition 4.3.16. □

4.4.2. *Proofs of the main theorems.* In this section we deduce the main theorems of the introduction.

Proof of Theorem 2. Recall that we assumed in Theorem 2 that $\pi_1(G)_I$ is torsion free, which implies that all parahoric subgroups of $G(\mathbb{Q}_p)$ are connected by Lemma 2.2.2. Part 1 of Theorem 2 is Theorem A.4.5.

Part 2 of the theorem for an Iwahori subgroup follows from Theorem 3.3.1 in combination with Theorem 4.4.1. To apply Theorem 4.4.1 we need to verify that either $\mathbf{Sh}_U(G,X)$ is proper or Conjecture 4.3.1 holds. In the statement of Theorem 2 we are assuming that either $\mathbf{Sh}_U(G,X)$ is proper or that $G_{\mathbb{Q}_p}$ is unramified. Now we recall that Conjecture 4.3.1 holds if $G_{\mathbb{Q}_p}$ is unramified by [53, Proposition 6.20] and the main result of [1], see Remark 4.3.2. Part 2 of the theorem for an arbitrary parahoric subgroup follows from the case of an Iwahori subgroup by [55, Proposition 7.8]. □

Proof of Theorem 1. Recall that we assumed in Theorem 2 that $\pi_1(G)_I$ is torsion free, which implies that all parahoric subgroups of $G(\mathbb{Q}_p)$ are connected by Lemma 2.2.2. Theorem 1 is therefore a direct consequence of Corollary 3.2.1. □

Proof of Theorem 3. By [55, Theorem 8.1.(ii)], uniformisation of isogeny classes, as proved in Theorem 2, implies that the He–Rapoport axioms hold. □

Proof of Theorem 4. This follows from Theorem 4.5.2 below by specialising to the case that G^{ad} is $\mathbb{Q}$-simple. Note that Theorem 4.5.2 has the assumption that either $\mathbf{Sh}_U(G,X)$ is proper or Conjecture 4.3.1 holds, which is true if either $\mathbf{Sh}_U(G,X)$ is proper or if $G_{\mathbb{Q}_p}$ is unramified, see the proof of Theorem 2 above. □

4.5. **Consequences for irreducible components.** In this section we prove a generalisation of Theorem 4. Before we can state it, we need to introduce some notation. Let $G^{\mathrm{ad}} = \prod_i G_i$ be the decomposition of G^{ad} into simple groups over $\mathbb{Q}$ and consider the induced maps of Kottwitz sets

$$B(G_{\mathbb{Q}_p}) \to B(G^{\mathrm{ad}}_{\mathbb{Q}_p}) \to \prod_i B(G_{i,\mathbb{Q}_p}).$$

Definition 4.5.1 (Definition 5.3.2 of [30])**.** An element $[b] \in B(G)$ is called $\mathbb{Q}$-*non-basic* if the image of $[b]$ in $B(G_{i,\mathbb{Q}_p})$ is non-basic for all i. A Newton stratum $\mathrm{Sh}_{G,K,[b],U^p}$ is called $\mathbb{Q}$-*non-basic* if $[b]$ is $\mathbb{Q}$-non-basic.

Recall that $K \subset \mathbb{S}$ corresponds to a very special parahoric.

Theorem 4.5.2. *Let $w \in {}^K\mathrm{Adm}(\{\mu\})$ such that the EKOR stratum $\mathrm{Sh}_{G,K,U^p}\{w\}$ intersects a $\mathbb{Q}$-non-basic Newton stratum. If either $\mathbf{Sh}_U(G,X)$ is proper or Conjecture 4.3.1 holds, then*

$$\mathrm{Sh}_{G,K,U^p}\{w\} \to \mathrm{Sh}_{G,K,U^p}$$

induces a bijection on π_0.

We start by proving a lemma.

Lemma 4.5.3. *For $w \in {}^K\mathrm{Adm}(\{\mu\})$, viewed as an element of $\mathrm{Adm}(\{\mu\})$ via ${}^K\mathrm{Adm}(\{\mu\}) \subset \mathrm{Adm}(\{\mu\})$, the forgetful map $\mathrm{Sh}_{G,\emptyset,U^p}(w) \to \mathrm{Sh}_{G,K,U^p}$ factors through $\mathrm{Sh}_{G,K,U^p}\{w\}$, via a surjective map $\mathrm{Sh}_{G,\emptyset,U^p}(w) \to \mathrm{Sh}_{G,K,U^p}\{w\}$.*

Proof. The factorisation is [50, Theorem 5.4.5.(3)] and the surjectivity is proved there under the assumption that Axiom 4(c) of [16] holds, which is true by Theorem 3. □

Proof of Theorem 4.5.2. We will prove that if $w \in \mathrm{Adm}(\{\mu\})$ such that $\mathrm{Sh}_{G,\emptyset,U^p}(w)$ intersects a $\mathbb{Q}$-non-basic Newton stratum, then the natural map $\pi_0(\mathrm{Sh}_{G,\emptyset,U^p}(w)) \to \pi_0(\mathrm{Sh}_{G,K,U^p})$ is a bijection. By Lemma 4.5.3, this will imply Theorem 4.5.2.

Step 1: We first deal with the case of σ-straight $w \in \mathrm{Adm}(\{\mu\})$. Then $\mathrm{Sh}_{G,\emptyset,U^p}(w)$ is contained in a unique Newton stratum $\mathrm{Sh}_{G,\emptyset,[b],U^p}$, see [50, Theorem 1.3.5], which by assumption is $\mathbb{Q}$-non-basic. We deduce from Theorem 4.4.1, Corollary 4.2.3 and Proposition 4.3.16 that for any finite set of primes Σ with $p \in \Sigma$, the group $G^{\mathrm{sc}}(\mathbb{A}_f^{\Sigma})$ acts transitively on the fibers of

$$\pi_0(\mathrm{Sh}_{G,\emptyset}(w)) \to \pi_0(\mathrm{Sh}_{G,K}).$$

By Lemma 4.3.7, there exists $y \in {}^K\mathrm{Adm}(\{\mu\})$ such that the natural map $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(w) \to \mathrm{Sh}_{G,K,U^p}$ factors via a finite étale map $\widehat{\mathrm{Sh}}_{G,\emptyset,U^p}(w) \to \mathrm{Sh}_{G,K,U^p}\{y\}$. We want to apply [23, Theorem 3.4.1] to the $G(\mathbb{A}_f^p)$-equivariant finite étale cover $\mathrm{Sh}_{G,\emptyset}(w) \to \mathrm{Sh}_{G,K}\{y\}$. Note that [23, Hypothesis 2.3.1] follows from Theorem 1, see [23, Remark 2.3.3]. Moreover, [23, Hypothesis 3.4.1] of loc. cit. is satisfied since Sh_{G,K,U^p} is locally integral because $\mathcal{G}_K$ is very special, see [25, Corollary 4.6.26]. The assumption that every connected component of $\mathrm{Sh}_{G,K,U^p}\{y\}$ intersects a $\mathbb{Q}$-non-basic Newton stratum holds since $\mathrm{Sh}_{G,\emptyset,U^p}\{y\}$ is contained in a single $\mathbb{Q}$-non-basic Newton stratum since $\mathrm{Sh}_{G,\emptyset,U^p}(w)$ is.

Therefore the assumptions of [23, Theorem 3.4.1] are satisfied and we conclude that if Σ contains all the primes ℓ where $G^{\mathrm{ad}}_{\mathbb{Q}_\ell}$ has a compact factor, then $G^{\mathrm{sc}}(\mathbb{A}_f^{\Sigma})$ acts trivially on the fibers of $\pi_0(\mathrm{Sh}_{G,\emptyset}(w)) \to \pi_0(\mathrm{Sh}_{G,K})$. Since it also acts transitively on these fibers by Proposition 4.3.16 and since the map $\pi_0(\mathrm{Sh}_{G,\emptyset}(w)) \to \pi_0(\mathrm{Sh}_{G,K})$ is surjective by Corollary 3.4.7, we deduce that $\pi_0(\mathrm{Sh}_{G,\emptyset}(w)) \to \pi_0(\mathrm{Sh}_{G,K})$ is a bijection.

Step 2: For general $w \in \mathrm{Adm}(\{\mu\})$ intersecting a $\mathbb{Q}$-non-basic Newton stratum $\mathrm{Sh}_{G,\emptyset,[b],U^p}$, there is a σ-straight element $w' \leq w$ with $\mathrm{Sh}_{G,\emptyset,U^p}(w') \cap \mathrm{Sh}_{G,\emptyset,[b],U^p}$ non-empty; this follows from [19, Theorem 4.1] as in the proof of Lemma 4.3.6. It follows from our assumptions that every connected component V of $\mathrm{Sh}_{G,\emptyset,U^p}(\leq w)$ intersects $\mathrm{Sh}_{G,\emptyset,U^p}(\tau)$. Thus the intersection $V \cap \mathrm{Sh}_{G,\emptyset,U^p}(\leq w')$ is non-empty, and it is therefore a union of connected components of $\mathrm{Sh}_{G,\emptyset,U^p}(\leq w')$. Hence $V \cap \mathrm{Sh}_{G,\emptyset,U^p}(\leq w')$ is equidimensional of dimension $\ell(w')$ and must therefore intersect $\mathrm{Sh}_{G,\emptyset,U^p}(w')$. We see that the natural map $\mathrm{Sh}_{G,\emptyset,U^p}(w') \to \mathrm{Sh}_{G,\emptyset,U^p}(\leq w)$ induces a surjective map on π_0. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Sh}_{G,\emptyset}(w') & \longrightarrow & \pi_0(\mathrm{Sh}_{G,\emptyset}(\leq w) \\ & \searrow \quad \swarrow & \\ & \pi_0(\mathrm{Sh}_{G,K}). & \end{array}$$

The right diagonal map is surjective by Corollary 3.4.7, the horizontal map is surjective by the discussion above and the left diagonal arrow is a bijection by step 1. It follows that $\pi_0(\mathrm{Sh}_{G,\emptyset}(\leq w)) \to \pi_0(\mathrm{Sh}_{G,K})$ is a bijection and since $\pi_0(\mathrm{Sh}_{G,\emptyset}(w)) \to \pi_0(\mathrm{Sh}_{G,\emptyset}(\leq w))$ is a bijection by the local integrality of $\pi_0(\mathrm{Sh}_{G,\emptyset}(\leq w))$, see Corollary 4.2.3, we are done. □

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Appendix A. Connected components of affine Deligne–Lusztig varieties with very special level structure, by Rong Zhou

As explained in the introduction, proving uniformisation of isogeny classes in Shimura varieties of Hodge type with parahoric level is closely related to the problem of understanding connected components of affine Deligne–Lusztig varieties with parahoric level. In this Appendix, we study the connected components of affine Deligne–Lusztig varieties with very special level structure and prove for instance that $J_b(\mathbb{Q}_p)$ acts transitively on these connected components. These results will be used in Section A.4 to prove uniformisation of isogeny classes in Shimura varieties of Hodge type with very special level.

A.1. **The main result.** We follow the notation of Section 2. Thus G is a reductive group over $\mathbb{Q}_p$ and $\{\mu\}$ is a geometric conjugacy class of cocharacters of $G_{\overline{\mathbb{Q}}_p}$. We assume that G is quasi-split, and we let $\mathcal{I}$ be the Iwahori group scheme corresponding to a σ-stable alcove $\mathfrak{a}$ in the building for G. We fix $\mathcal{G}$ a very special standard parahoric group scheme for G. Then $\mathcal{G}$ corresponds to a σ-stable special point $\mathfrak{s}$ lying in the closure of $\mathfrak{a}$ and we write $K \subset \mathbb{S}$ for the subset of simple affine reflections which preserve $\mathfrak{s}$. The projection $\tilde{W} \to W_0$ induces an isomorphism $W_K \cong W_0$.

As explained in [55, §9], we have an identification

$$W_K \backslash \tilde{W} / W_K \cong X_*(T)_I^+.$$

By [16], there exists a reduced root system Σ (the échelonnage root system) such that

$$W_a \cong W(\Sigma) \ltimes Q^\vee(\Sigma),$$

where $W(\Sigma)$ (resp. $Q^\vee(\Sigma)$) is the Weyl group (resp. coroot lattice) of Σ. We define a partial order $\preccurlyeq$ on $X_*(T)_I^+$ by setting $\lambda \preccurlyeq \lambda'$ if $\lambda - \lambda'$ can be written as a sum of positive coroots in $Q^\vee(\Sigma)$ with positive integral coefficients. Then by [35, pp. 210], the Bruhat order on $W_K \backslash \tilde{W} / W_K$ agrees with the partial order $\preccurlyeq$. It follows that for $\mu \in X_*(T)_I^+$, we have

$$\mathrm{Adm}(\mu)_K = \{\lambda \in X_*(T)_I^+ | \lambda \preccurlyeq \mu\}.$$

Let $b \in G(\breve{\mathbb{Q}}_p)$ such that $[b] \in B(G, \{\mu\})$. We have the affine Deligne–Lusztig variety $X(\mu, b)_K$ defined in Section 2.4.2. We also set

$$\mathrm{Adm}(\mu)^K := W_K \mathrm{Adm}(\mu) W_K \subset \tilde{W}$$

and define

$$X(\mu, b)^K := \bigcup_{w \in \mathrm{Adm}(\mu)^K} X_w(b),$$

which is a locally closed subscheme of the Witt vector affine flag variety $\mathrm{Gr}_{\mathcal{I}}$. Then there is a natural map

$$X(\mu,b)^K \to X(\mu,b)_K \tag{A.1.1}$$

which is equivariant for the action of the σ-centraliser group $J_b(\mathbb{Q}_p)$. In fact, (A.1.1) is a fibration with connected fibers and hence induces a $J_b(\mathbb{Q}_p)$-equivariant bijection

$$\pi_0(X(\mu,b)^K) \xrightarrow{\sim} \pi_0(X(\mu,b)_K). \tag{A.1.2}$$

A.1.1. Recall from e.g [28, Section 1.1.2] that associated to $[b] \in B(G)$ there is a *Newton cocharacter* $\overline{\nu}_{[b]}$. Let M denote the centraliser of $\overline{\nu}_{[b]}$ and we fix a representative b of $[b]$ such that $\nu_{[b]} = \overline{\nu}_{[b]}$. The existence of such a representative follows from the same argument as in [6, Lemma 2.5.2] which also shows that $b \in M(\breve{\mathbb{Q}}_p)$. Then b is a basic element of M, in other words $\overline{\nu}_{[b]}$ is central in M.

We use a subscript M to denote the corresponding objects for M. Thus $\tilde{W}_M$ (resp. $W_{a,M}$) denotes the Iwahori–Weyl group (resp. affine Weyl group) for M. The intersection $M(\breve{\mathbb{Q}}_p) \cap \mathcal{G}(\breve{\mathbb{Z}}_p)$ arises as the $\breve{\mathbb{Z}}_p$-points of a very special parahoric group scheme $\mathcal{M}$ for M, which is standard for the alcove $\mathfrak{a}_M$ for M determined by $\mathfrak{a}$. We write $\mathcal{I}_M$ for the Iwahori group scheme of M determined by $\mathfrak{a}_M$ and we let $K_M \subset \mathbb{S}_M$ denote the subset of simple affine reflections for M corresponding to $\mathcal{M}$. We let Σ_M denote the échelonnage root system for M so that

$$W_{a,M} \cong W(\Sigma_M) \ltimes Q^\vee(\Sigma_M).$$

For $x \in \pi_1(M)_I$, we write $\tau_x \in \Omega_M$ for the corresponding length 0 element, and we write $\tau_x = t^{\mu_x} w_x$ for a unique $w_x \in W_K$. Then the map $x \mapsto \mu_x$ induces a bijection

$$\pi_1(M)_I \cong \{\lambda \in X_*(T)_I | \lambda \text{ is } M\text{-dominant and } M\text{-minuscule}\}, \tag{A.1.3}$$

here M-minuscule means minuscule with respect to the root system Σ_M. We define the set

$$I_{\mu,b,M} := \{x \in \pi_1(M)_I | \kappa_M(b) = x, \mu_x \preccurlyeq \mu\}.$$

Via the bijection (A.1.3), we also consider $I_{\mu,b,M}$ as a subset of the set of M-minuscule and M-dominant elements in $X_*(T)_I$. For each $\lambda \in I_{\mu,b,M}$, we have the affine Deligne–Lusztig variety $X^M(\lambda,b)_{K_M}$ for the group M. It is a closed subscheme of the partial affine flag variety for M with respect to the parahoric subgroup $\mathcal{M}$, and its $\overline{\mathbb{F}}_p$-points are given by

$$\{m \in M(\breve{\mathbb{Q}}_p)/\mathcal{M}(\breve{\mathbb{Z}}_p) | m^{-1} b \sigma(m) \in \mathcal{M}(\breve{\mathbb{Z}}_p) \dot{t}^{\lambda'} \mathcal{M}(\breve{\mathbb{Z}}_p), \lambda' \preccurlyeq_M \lambda\}.$$

It is equipped with a natural map

$$X^M(\lambda,b)_{K_M} \to X(\mu,b)_K \tag{A.1.4}$$

which is equivariant for the action of the σ-centraliser group $J_b(\mathbb{Q}_p)$.

A.1.2. Our main theorem on the connected components of affine Deligne–Lusztig varieties is the following.

Theorem A.1.3. *$J_b(\mathbb{Q}_p)$ acts transitively on $\pi_0(X(\mu,b)_K)$. In particular, for any $\lambda \in I_{\mu,b,M}$ the map* (A.1.4) *induces a surjection*

$$\pi_0(X^M(\lambda,b)_{K_M}) \to \pi_0(X(\mu,b)_K).$$

Remark A.1.4. The theorem is stated for G a quasi-split reductive group over $\mathbb{Q}_p$. However, the result makes sense for general quasi-split groups over any local field F and can be proved in exactly the same way.

A.1.5. We follow the strategy of [6] and [39] where this result was proved for unramified groups G. The result follows from the following two propositions.

Proposition A.1.6. The natural map

$$\coprod_{\lambda \in I_{\mu,b,M}} X^M(\lambda, b)_{K_M} \to X(\mu, b)_K$$

induces a surjection

$$\coprod_{\lambda \in I_{\mu,b,M}} \pi_0(X^M(\lambda, b)_{K_M}) \to \pi_0(X(\mu, b)_K).$$

Proposition A.1.7. Let $\lambda \in I_{\mu,b,M}$. The image of the natural map

$$\pi_0(X^M(\lambda, b)_{K_M}) \to \pi_0(X(\mu, b)_K)$$

does not depend on the choice of $\lambda \in I_{\mu,b,M}$.

Proof of Theorem A.1.3. Fix $\lambda \in I_{\mu,b,M}$. By Proposition A.1.6 and Proposition A.1.7, the map

$$\pi_0(X^M(\lambda, b)_{K_M}) \to \pi_0(X(\mu, b)_K)$$

is surjective. By [19, Theorem 4.1 and Theorem 5.1], $J_b(\mathbb{Q}_p)$ acts transitively on $\pi_0(X^M(\lambda, b)_{K_M})$, and hence on $\pi_0(X(\mu, b)_K)$.

□

A.1.8. We now proceed to prove the two propositions. Note that by a standard reduction (see [19, Section 6]), it suffices to prove the propositions when G is adjoint and $\mathbb{Q}_p$-simple. We may and do assume this from now on.

A.2. **Proof of Proposition A.1.6.**

A.2.1. In the case of unramified groups, Proposition A.1.6 is [6, Proposition 3.4.1]. Here we prove the general case using a different method based on the Deligne–Lusztig reduction method for affine Deligne–Lusztig varieties in the affine flag variety.

We begin with some preliminaries regarding σ-conjugacy classes in Iwahori–Weyl groups. For any element $w \in \tilde{W}$, we let n be a sufficiently divisible integer such that σ^n acts trivially on $\tilde{W}$ and $w\sigma(w)\dots\sigma^{n-1}(w) = t^\lambda$ for some $\lambda \in X_*(T)_I$. We set $\nu_w := \frac{t^\lambda}{n} \in X_*(T)_{I,\mathbb{Q}}$ and $\overline{\nu}_w \in X_*(T)^+_{I,\mathbb{Q}}$ for the dominant representative of ν_w. We let $\kappa(w) \in \pi_1(G)_\Gamma$ denote the image of w under the projection $\tilde{W} \to \pi_1(G)_I \to \pi_1(G)_\Gamma$. We write $B(\tilde{W}, \sigma)$ for the set of σ-conjugacy classes in $\tilde{W}$. Then $w \mapsto [\dot{w}]$ induces a well-defined map $\Psi : B(\tilde{W}, \sigma) \to B(G)$ and we have a commutative diagram (see [17, Section 3.3, Theorem 3.5]):

$$\begin{array}{ccc} B(\tilde{W}, \tilde{s}) & \xrightarrow{\quad\Psi\quad} & B(G) \\ & {\scriptstyle (\overline{\nu},\kappa)} \searrow \quad \swarrow {\scriptstyle (\overline{\nu},\kappa)} & \\ & (X_*(T)^+_{I,\mathbb{Q}})^\sigma \times \pi_1(G)_\Gamma & \end{array} .$$

A.2.2. We will need the following lemma. We write $J \subset K \cong \mathbb{S}_0$ for the subset corresponding to M. Recall that an element $w \in \tilde{W}$ is said to be σ-straight if

$$n\ell(w) = \ell(w\sigma(w)\dots\sigma^{n-1}(w))$$

for all n.

Lemma A.2.3. *Let $w \in \tilde{W}$ be σ-straight such that $\dot{w} \in [b]$; in particular $\overline{\nu}_b = \overline{\nu}_w$. Let $u \in {}^JW_0$ such that $u(\nu_w) = \overline{\nu}_b$ and set $w_\sharp := uw\sigma(u)^{-1}$. Then $w_\sharp \in \Omega_M$.*

Proof. It suffices to show that $w_\sharp \in W_M$ and

$$w_\sharp \mathcal{I}_M(\breve{\mathbb{Z}}_p) w_\sharp^{-1} = \mathcal{I}_M(\breve{\mathbb{Z}}_p).$$

The first statement follows since $w_\sharp(\overline{\nu}_w) = \overline{\nu}_w$. By [19, Theorem 5.2], the element w is (ν_w, σ)-fundamental. Thus

$$w\sigma(\mathcal{I}_{M_{\nu_w}}(\breve{\mathbb{Z}}_p))w^{-1} = \mathcal{I}_{M_{\nu_w}}(\breve{\mathbb{Z}}_p)$$

where M_{ν_w} is the centraliser of ν_w and $\mathcal{I}_{M_{\nu_w}}(\breve{\mathbb{Z}}_p) := \mathcal{I}(\breve{\mathbb{Z}}_p) \cap M_{\nu_w}(\breve{\mathbb{Q}}_p)$. Since $u \in {}^JW_0$, we have $u\mathcal{I}_{M_{\nu_w}}(\breve{\mathbb{Z}}_p)u^{-1} = \mathcal{I}_M(\breve{\mathbb{Z}}_p)$. It follows that

$$\begin{aligned} w_\sharp \mathcal{I}_M(\breve{\mathbb{Z}}_p) w_\sharp^{-1} &= w_\sharp \sigma(\mathcal{I}_M(\breve{\mathbb{Z}}_p)) w_\sharp^{-1} \\ &= uw\sigma(\mathcal{I}_{M_{\nu_w}}(\breve{\mathbb{Z}}_p))w^{-1}u^{-1} \\ &= \mathcal{I}_M(\breve{\mathbb{Z}}_p). \end{aligned}$$

as desired. □

Proof of Proposition A.1.6. By (A.1.2), it suffices to show the natural map

$$\coprod_{\lambda \in I_{\mu,b,M}} X^M(\lambda, b)^{K_M} \to X(\mu, b)^K$$

induces a surjection

$$\coprod_{\lambda \in I_{\mu,b,M}} \pi_0(X^M(\lambda, b)^{K_M}) \to \pi_0(X(\mu, b)^K).$$

Let Y be a connected component of $X(\mu, b)^K$. Then by [19, Theorem 4.1], there exists a σ-straight element $w \in \mathrm{Adm}(\mu)^K$, such that $Y \cap X_w(b) \neq \emptyset$. Let $w_\sharp$ denote the element constructed in Lemma A.2.3 and $u \in {}^JW_0$ such that $uw\sigma(u) = w_\sharp$. Then we claim that $[b]_M = [\dot{w}_\sharp]_M \in B(M)$. Indeed, we have $\nu_{w_\sharp} = \overline{\nu}_w = \overline{\nu}_b$. Therefore the image of $[b]_M$ and $[\dot{w}_\sharp]_M$ in $\pi_1(M)_I$ coincide up to torsion. On the other hand, the images of $[b]_M$ and $[\dot{w}_\sharp]_M$ in $\pi_1(G)_I$ coincide and $\ker(\pi_1(M)_I \to \pi_1(G)_I)$ is torsion free. It follows that $\kappa_M([b]_M) = \kappa_M([\dot{w}_\sharp]_M)$ and hence $[b]_M = [\dot{w}_\sharp]_M$. Thus we may replace b by $\dot{w}_\sharp$.

We will show that $Y \cap X^M_{w_\sharp}(\dot{w}_\sharp) \neq \emptyset$. Since $w_\sharp \in \mathrm{Adm}^M(\lambda)^{K_M}$, where $\lambda \in I_{\mu,b,M}$ corresponds to the image of $w_\sharp$ in $\pi_1(M)_I$, it follows that $X_{w_\sharp}(\dot{w}_\sharp) \subset X^M(\lambda, \dot{w}_\sharp)^{K_M}$; this implies the proposition.

For any affine root α, we let $\mathcal{U}_{-\alpha_i}$ denote the affine root subgroup corresponding to $-\alpha_i$ over $\breve{\mathbb{Z}}_p$. By [5, Section 4.3.2, 4.3.5 and 4.3.7], $\mathcal{U}_{-\alpha_i}$ is the group scheme associated to a finite free $\breve{\mathbb{Z}}_p$-module. For any $\epsilon > 0$, we let $\mathcal{U}_{-\alpha_i+\epsilon}(\breve{\mathbb{Z}}_p)$ be the subgroup of $\mathcal{U}_{-\alpha_i}(\breve{\mathbb{Z}}_p)$ corresponding to the affine function $-\alpha + \epsilon$. Similarly, we write $\mathcal{U}_{-\alpha_i+}(\breve{\mathbb{Z}}_p)$ for the union of $\mathcal{U}_{-\alpha_i+\epsilon}(\breve{\mathbb{Z}}_p)$ over all $\epsilon > 0$. As the notation suggests, these arise as the $\breve{\mathbb{Z}}_p$-points of group schemes $\mathcal{U}_{-\alpha_i+\epsilon}$ and $\mathcal{U}_{-\alpha_i+\epsilon}$ over $\breve{\mathbb{Z}}_p$, and the quotient $\mathcal{U}_{-\alpha_i}(\breve{\mathbb{Z}}_p)/\mathcal{U}_{-\alpha_i+}(\breve{\mathbb{Z}}_p)$ is a 1-dimensional vector space over k. We choose a group scheme homomorphism

$$f : \mathbb{G}_a \to \mathcal{U}_{-\alpha}$$

which lifts the map $k \cong \mathcal{U}_{-\alpha_i}(\breve{\mathbb{Z}}_p)/\mathcal{U}_{-\alpha_i+\epsilon}(\breve{\mathbb{Z}}_p)$.

For R a perfect k-algebra and $a \in R$, the map

$$h_\alpha : R \to \mathcal{U}_{-\alpha}(W(R))$$
$$a \mapsto f([a])$$

where $[a] \in W(R)$ is the Teichmüller lift of a, induces a k-scheme morphism

$$h_\alpha : \mathbb{A}^{1,\mathrm{perf}} \to L^+\mathcal{U}_{-\alpha},$$

where $\mathbb{A}^{1,\mathrm{perf}}$ denotes the perfection of $\mathbb{A}^1$ over k. The induced morphism $\mathbb{A}^{1,\mathrm{perf}} \to \mathrm{Gr}_{\mathcal{I}}$ extends to a morphism $\mathbb{P}^{1,\mathrm{perf}} \to \mathrm{Gr}_{\mathcal{I}}$ also denoted h_α. Then we have

$$h_\alpha(\infty) = \dot{s}_\alpha \mathcal{I} \in \mathrm{Gr}_{\mathcal{I}}. \tag{A.2.1}$$

Let $g_0\mathcal{I} \in Y \cap X_w(\dot{w}_\sharp)$ with $g_0 \in G(\breve{\mathbb{Q}}_p)$. By [19, Theorem. 5.2], we may choose g_0 such that $g_0^{-1}\dot{w}_\sharp g_0 = \dot{w}$. Let $s_n \dots s_1$ be a reduced word decomposition for u (note that $s_i \in K$) and we write u_i for the element $s_1 \dots s_i \in \tilde{W}$ and $u_0 = e$. We write $g_i \in G(\breve{\mathbb{Q}}_p)$for the element $g_0\dot{u}_i$. We will prove by induction that $g_i\mathcal{I} \in Y$ for $i = 0, 1, \dots, n$; clearly this is true for $i = 0$.

Assume $g_i \in Y$ and we let α_{i+1} denote the positive affine root corresponding to s_{i+1}. We consider the map

$$g := g_i h_{\alpha_{i+1}} : \mathbb{P}^{1,\mathrm{perf}} \to \mathrm{Gr}_{\mathcal{I}}.$$

Since $\mathcal{U}_{-\alpha_{i+1}} \subset \mathcal{I}\dot{s}_{i+1}\mathcal{I}$, for any $s \in \mathbb{A}^{1,\mathrm{perf}}(k)$ we have

$$\begin{aligned} g(s)^{-1}\dot{w}_\sharp\sigma(g(s)) &= h_{\alpha_{i+1}}(s)^{-1} g_i^{-1}\dot{w}_\sharp\sigma(g_i)\sigma(h_{\alpha_{i+1}}(s)) \\ &\in \mathcal{I}\dot{s}_{\alpha_{i+1}}\mathcal{I}\dot{w}_i\mathcal{I}\sigma(\dot{s}_{\alpha_{i+1}})\mathcal{I} \\ &\subset \bigcup_{x\in A} \mathcal{I}\dot{x}\mathcal{I} \end{aligned}$$

where $w_i := u_i^{-1} w_\sharp \sigma(u_i) \in \mathrm{Adm}(\mu)^K$ and $A \subset \tilde{W}$ is the subset

$$A = \{w_i, s_{i+1}w_i, w_i\sigma(s_{i+1}), s_{i+1}w_i\sigma(s_{i+1})\}.$$

Since $\mathrm{Adm}(\mu)^K$ is closed under left and right multiplication by W_K, we have $A \subset \mathrm{Adm}(\mu)^K$, and hence

$$g(s)^{-1}\dot{w}^\sharp\sigma(g(s)) \in \bigcup_{v\in\mathrm{Adm}(\mu)^K} \mathcal{I}\dot{v}\mathcal{I}$$

for any $s \in \mathbb{P}^{1,\mathrm{perf}}(k)$. Moreover we have $g(0) = g_i\mathcal{I}$ and $g(\infty) = g_{i+1}\mathcal{I}$, where the latter equality follows from (A.2.1). Thus the image of g is a curve in $X(\mu, b)^K$ which connects $g_i\mathcal{I}$ and $g_{i+1}\mathcal{I}$ and hence $g_{i+1}\mathcal{I} \in Y$. Then by definition g_n, $g_n\mathcal{I}$ lies in the image of $X^M_{w_\sharp}(\dot{w}_\sharp)$ as desired. □

A.3. **Proof of Proposition A.1.7.**

A.3.1. When G is unramified, this proposition follows from the proof of [6, Proposition 4.1.12] when μ is minuscule; the general case is proved in [39, Proposition 5.1]. The main input is the construction of explicit curves in $X(\mu, b)_K$ which connect points in $X^M(\lambda, b)_{K_M}$ and $X^M(\lambda', b)_{K_M}$ for $\lambda \neq \lambda' \in I_{\mu,b,M}$. The construction of these curves relied on certain combinatorial results concerning the root system for G. The exact same method of proof works in our setting; however, there are a few subtleties which we now explain.

First, the explicit curves were constructed in [6] and [39] using root subgroups of $G_{\breve{\mathbb{Q}}_p}$ which are all isomorphic to $\mathbb{G}_a$ when the group is unramified. In general, the root subgroups are more complicated and thus one needs to be more careful. However, we are still able to give a uniform construction of the curves that we need.

Secondly, we need to generalise the combinatorial results to general quasi-split groups G. It turns out there is a systematic way to deduce these combinatorial results for quasi-split G from the case of unramified groups which we now explain.

A.3.2. Recall we have assumed G is adjoint and simple. As in [17, §7.2], see [56, Proof of Theorem A.3.1] for an explicit construction, there is an unramified adjoint group H over $\mathbb{Q}_p$ such that the pair (W', σ') consisting of the Iwahori–Weyl group for G' and the action of Frobenius is identified with the pair (W, σ). Moreover the échelonnage root system Σ is identified with the absolute root system Σ' for G', and we have an isomorphism

$$\pi_1(G)_I \cong \pi_1(G')$$
$$X_*(T)_I \cong X_*(T'),$$

where T' is a suitable maximal $\breve{\mathbb{Q}}_p$-split torus of G'.

We use a superscript $'$ to denote the corresponding objects for G'. Then $\mathcal{G}$ determines a hyperspecial subgroup $\mathcal{G}'$ for G' and we write $K' \subset \mathbb{S}'$ for the corresponding subset of simple reflections. Then M determines a Levi subgroup M' of G' and hence a subset $J' \subset K'$. It follows that the combinatorial data

$$(\Sigma, X_*(T)_I, \sigma, J, \mu, \kappa_M(b)) \tag{A.3.1}$$

is identified with the corresponding data for G'. Thus any result which only depends on the data ((A.3.1)), can be reduced to the case of unramified groups. The combinatorial results that we need are already proved in the case of unramified groups in [39] and [6]. We therefore take the convention that whenever we need certain results which depend on the data ((A.3.1)), we will refer to the relevant result in [6] or [39].

A.3.3. We now proceed with the proof of Proposition A.1.7. Let $x, x' \in \pi_1(M)_I$. We write $x \overset{(\alpha,r)}{\to} x'$ for some $\alpha \in \Sigma$ and $r \in \mathbb{N}$ if $x - x' = \alpha^\vee - \sigma^r(\alpha^\vee)$ and

$$\mu_x, \mu_{x+\alpha^\vee}, \mu_{x-\sigma^r(\alpha^\vee)}, \mu_{x'} \preccurlyeq \mu.$$

We write $x \overset{(\alpha,r)}{\rightarrowtail} x'$ if $x \overset{(\alpha,r)}{\to} x'$ and neither

$$x \overset{(\alpha,i)}{\to} x + \alpha^\vee - \sigma^i(\alpha^\vee) \overset{(\sigma^i(\alpha),r-i)}{\to} x'$$

nor

$$x \overset{(\sigma^i(\alpha),r-i)}{\to} x + \sigma^i(\alpha^\vee) - \sigma^r(\alpha^\vee) \overset{(\alpha,i)}{\to} x'$$

for any $i \in [1, r-1]$.

We let

$$\langle\ ,\ \rangle : (Q(\Sigma) \otimes_{\mathbb{Z}} \mathbb{R}) \times (X_*(T)_I \otimes_{\mathbb{Z}} \mathbb{R}) \to \mathbb{R}$$

be the natural pairing, where $Q(\Sigma)$ is the root lattice of Σ. For any element $\alpha \in \Sigma$, we write $\mathcal{O}_\alpha$ for the σ-orbit of α. We let h denote the number of connected components of the Dynkin diagram of G over $\breve{\mathbb{Q}}_p$; then we have $\#\mathcal{O}_\alpha \in \{h, 2h, 3h\}$.

Lemma A.3.4 ([39, Lemma 7.7])**.** *Let $x \neq x' \in I_{\mu,b,M}$. Then there exists $x_j \in \pi_1(M)_I$, $\alpha_j \in \Sigma - \Sigma_M$ and $r_j \in \mathbb{N}$ for $j \in [0, m-1]$ such that*

(1) $\alpha_j^\vee$ is M-dominant and M-minuscule.
(2) $r_j \in [1, h]$ if $\#\mathcal{O}_{\alpha_j} \in \{h, 2h\}$ and $r_j \in [1, 2h-1]$ if $\#\mathcal{O}_{\alpha_j} = 3h$.
(3) $x_0 = x, x_m = x'$ and we have $x_j \overset{(\alpha_j,r_j)}{\to} x_{j+1}$ for $j \in [0, m-1]$.

Proof. This follows from [39, Lemma 7.7] by discussion in Section A.3.2 above. Note that in loc. cit., the result is stated for M a Levi subgroup such that b is superbasic in M. However, one checks that the same proof works for any M as long as $I_{\mu,b,M}$ contains a weakly dominant element. Here $\lambda \in X_*(T)_I$ is weakly dominant if $\langle \alpha, \lambda \rangle \geq -1$ for any positive root $\alpha \in \Sigma$. But as in [39, Lemma 4.1], any element $\lambda \in I_{\mu,b,M}$ is weakly dominant, so the result applies to our M.

□

A.3.5. We now construct certain curves inside LG which we will use to connect points in $X(\mu, b)_K$. Let $\alpha \in \Sigma$ be a root. Then α determines a relative root $\widetilde{\alpha}$ of G over $\breve{\mathbb{Q}}_p$ which we always take to be the short root. We let G_α denote the simply connected cover of the (semi-simple) group generated by $U_{\widetilde{\alpha}}$ and $U_{-\widetilde{\alpha}}$ and we write

$$i_\alpha : G_\alpha \to G$$

for the natural map. We let $\mathcal{G}_\alpha$ denote the very special parahoric of G_α such that $\mathcal{G}(\breve{\mathbb{Z}}_p) := G_\alpha(\breve{\mathbb{Q}}_p) \cap i_\alpha^{-1}(\mathcal{G}(\breve{\mathbb{Z}}_p))$.

If $\widetilde{\alpha}$ is not divisible, then we have an isomorphism

$$G_\alpha \cong \mathrm{Res}_{\breve{K}/\breve{\mathbb{Q}}_p} SL_2,$$

where $\breve{K}/\breve{\mathbb{Q}}_p$ is a finite extension. Then up to conjugacy $\mathcal{G}_\alpha$ is identified with the very special parahoric $SL_2(\mathcal{O}_{\breve{K}}) \subset G_\alpha(\breve{\mathbb{Q}}_p)$ and there is an isomorphism

$$f_{\widetilde{\alpha}} : \mathrm{Res}_{\breve{K}/\breve{\mathbb{Q}}_p} \mathbb{G}_a \xrightarrow{\sim} U_{\widetilde{\alpha}}.$$

If $\widetilde{\alpha}$ is divisible, then there is an isomorphism

$$G_\alpha \cong \mathrm{Res}_{\breve{K}/\breve{\mathbb{Q}}_p} SU_3,$$

where SU_3 is the special unitary group over $\breve{K}$ associated to a quadratic extension $\breve{K}'/\breve{K}$.

A.3.6. We recall the presentation of the $\breve{K}$-group SU_3 in [52, Example 1.15].

We let $\tau \in \mathrm{Gal}(\breve{K}'/\breve{K})$ denote the nontrivial element, and we consider the Hermitian form on $\breve{K}'^3$ given by

$$\langle (x_{-1}, x_0, x_1), (y_{-1}, y_0, y_1) \rangle = \tau(x_{-1})y_1 + \tau(x_0)y_0 + \tau(x_1)y_{-1}.$$

The group SU_3 is the special unitary group attached to this form. For $i = -1, 1$ and $c, d \in \breve{K}'$ such that $\tau(c)c + d + \tau(d) = 0$, we define

$$u_i(c, d) = I_3 + (g_{rs})$$

where I_3 is the identity matrix and (g_{rs}) is the matrix with entries $g_{-i,0} = -\tau(c)$, $g_{0,i} = c$, $g_{-i,i} = d$ and $g_{rs} = 0$ otherwise. The root subgroups are then given by

$$U_{\pm\widetilde{\alpha}}(\breve{K}) = \{u_{\pm 1}(c, d) | c, d \in \breve{K}', \tau(c)c + \tau(d) + d = 0\}$$

$$U_{\pm 2\widetilde{\alpha}}(\breve{K}) = \{u_{\pm 1}(0, d) | c, d \in \breve{K}', \tau(d) + d = 0\}.$$

We consider the very special parahoric $G_\alpha(\breve{F}) \cap GL_3(\mathcal{O}_{\breve{K}'})$ of $G_\alpha(\breve{F})$; we call this the standard parahoric. Let $\pi \in \breve{K}'$ be a uniformiser such that $\tau(\pi) = -\pi$ and let $s \in GL_3(\breve{K}')$ denote the element $\mathrm{diag}(\pi, 1, 1)$. Then the subgroup of $G_\alpha(\breve{\mathbb{Q}}_p)$ defined by

$$G_\alpha(\breve{\mathbb{Q}}_p) \cap s\breve{G}L_3(\mathcal{O}_{\breve{K}'})s^{-1}$$

is a very special parahoric subgroup of $G_\alpha(\breve{\mathbb{Q}}_p)$, which we shall call the non-standard parahoric. Up to conjugacy, these are the only very special parahorics of SU_3.

A.3.7. For $\alpha \in \Sigma$, we define a map $u_\alpha : \mathbb{A}^{1,\mathrm{perf}} \to LU_{\widetilde{\alpha}} \subset LG$ as follows. Let R be a perfect ring of characteristic p and $a \in R$ will denote an arbitrary element. We consider the following three separate cases.

(1) $G_\alpha \cong \mathrm{Res}_{\breve{K}/\breve{\mathbb{Q}}_p} SL_2$ and $\mathcal{G}_\alpha(\breve{\mathbb{Z}}_p) = SL_2(\mathcal{O}_{\breve{K}})$.
We define u_α to be the map induced by
$$a \mapsto i_\alpha(f_{\tilde{\alpha}}(\pi^{-1} \cdot [a])).$$

(2) $G_\alpha \cong \mathrm{Res}_{\breve{K}/\breve{\mathbb{Q}}_p} SU_3$ and $\mathcal{G}_\alpha$ is the standard parahoric subgroup.
We define u_α to be the map
$$a \mapsto i_\alpha(u_1(0, \pi^{-1} \cdot [a])).$$

(3) $G_\alpha \cong \mathrm{Res}_{\breve{K}/\breve{\mathbb{Q}}_p} SU_3$ and $\mathcal{G}_\alpha$ is the non-standard parahoric subgroup.
We define u_α to be the map
$$a \mapsto i_\alpha(u_1([a], \frac{[a]^2}{2})).$$

A.3.8. A calculation using the presentations of SL_2 or SU_3 above gives the following lemma (cf. [39, Lemma 7.14]).

Lemma A.3.9. *(1) Let $\lambda, \delta \in X_*(T)_I$ and $\alpha, \beta \in \Sigma$ such that $Q(\Sigma) \cap (\mathbb{Z}\alpha + \mathbb{Z}\beta)$ is of type A_2, $A_1 \times A_1$ or A_1 and such that*
$$\delta, \delta + \alpha^\vee, \delta - \beta^\vee, \delta + \alpha^\vee - \beta^\vee \preccurlyeq \lambda.$$
Then for all $y, z \in k$, we have
$$u_\alpha(z)\dot{t}^\delta u_\beta(y) \in \bigcup_{\lambda' \preccurlyeq \lambda} L^+\mathcal{G}\dot{t}^{\lambda'}L^+\mathcal{G}.$$

(2) Let $\alpha, \beta \in \Sigma$ and $\lambda \in X_*(T)_I$ such that $\langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle = -1$ and $\langle \alpha, \lambda \rangle \geq 2$. Then for any $y, z \in k$ we have
$$u_\beta(z)(\dot{t}^\lambda u_\alpha(y)\dot{t}^{-\lambda})u_\beta(-z) \in L^+\mathcal{G}$$

A.3.10. The following lemma is the analogue of [39, Lemma 7.8].

Lemma A.3.11. *Let $x, x' \in \pi_1(M)_I$, $\alpha \in \Sigma - \Sigma_M$ and $r \in \mathbb{N}$ such that*

(1) $\alpha^\vee$ is M-dominant and M-minuscule.
(2) $r \in [1, h]$ if $\#\mathcal{O}_\alpha \in \{h, 2h\}$ and $r \in [1, 2h-1]$ if $\#\mathcal{O}_\alpha = 3h$.
(3) $x \overset{(\alpha,r)}{\to} x'$.

Then for any $P \in X^M(\mu_x, b)_{K_M}$, there exists $P' \in X^M(\mu_{x'}, b)_{K_M}$ such that P and P' lie in the same connected component of $X(\mu, b)_K$ and we have
$$\kappa_M(P) - \kappa_M(P') = \sum_{i=0}^{r-1} \sigma^i(\alpha^\vee) \in \pi_1(M)_I.$$

Proof. As in [39, Lemma 7.5], we may assume that $x \overset{(\alpha,r)}{\rightarrowtail} x'$. Moreover, arguing as in [39, Lemma 7.15], it suffices to show that there exists $P \in X^M(\mu_x, b)_{K_M}$ and $P' \in X^M(\mu_{x'}, b)_{K_M}$ such that P and P' lie in the same connected component of $X(\mu, b)_K$ and we have $\kappa_M(P) - \kappa_M(P') = \sum_{i=0}^{r-1} \sigma^i(\alpha^\vee) \in \pi_1(M)_I$.

Let $b_x = \dot{t}^{\mu_x}\dot{w}_x$; then b_x is basic in M and since $\kappa_M(b) = \kappa_M(b_x) \in \pi_1(M)_\Gamma$, there exists $g_x \in M(\breve{F})$ such that $g_x^{-1} b \sigma(g_x) = b_x$. We define $P := g_x L^+\mathcal{M}$ so that $P \in X^M(\mu_x, b)_{K_M}$.

We first consider the case $r \in [1, h]$. For an element $g \in LG$, we write ${}^{b_x\sigma}g$ for the element $b_x\sigma(g)b_x^{-1}$. We define a map $u : \mathbb{A}^{1,\mathrm{perf}} \to \mathrm{Gr}_{\mathcal{G}}$ given by

$$u(z) = g_x u_\alpha(z)^{b_x\sigma} u_\alpha(z) \dots {}^{(b_x\sigma)^{r-1}} u_\alpha(z) L^+\mathcal{G}$$

Then by ind-projectivity of $\mathrm{Gr}_{\mathcal{G}}$, u extends to a map $g : \mathbb{P}^{1,\mathrm{perf}} \to \mathrm{Gr}_{\mathcal{G}}$. As in [39, Lemma 7.8], for any $z \in k$ we have

$$\begin{aligned} g(z)^{-1} b\sigma(g(z)) &\in L^+\mathcal{G} u_\alpha(-z) b_x \sigma \, {}^{(b_x\sigma)^{r-1}} u_\alpha(z) L^+\mathcal{G} \\ &= L^+\mathcal{G} u_\alpha(-z) \dot{t}^{\mu_x} u_{w_x\sigma^r(\alpha)}(c\sigma^r(z)) L^+\mathcal{G} \end{aligned}$$

for some $c \in k^\times$. Here we use [39, Corollary 7.12], which shows that $w_x\sigma^i(\alpha) = \sigma^i(\alpha)$ and $\langle \sigma^i(\alpha), \mu_x \rangle = 0$ for $i \in [1, r-1]$. By [6, Lemma 4.4.5], we have

$$\mu_x + \alpha^\vee, \mu_x - w_x(\sigma^r(\alpha^\vee)), \mu_x + \alpha^\vee - w_x(\sigma^r(\alpha^\vee)) \preccurlyeq \mu.$$

Thus by Lemma A.3.9 (1), we have

$$g(z)^{-1} b\sigma(g(z)) \in \bigcup_{\mu' \preccurlyeq \mu} L^+\mathcal{G} \dot{t}^{\mu'} L^+\mathcal{G}$$

and hence g factors through $X(\mu, b)_K$. Moreover one computes that

$$P' := g(\infty) = g_x \dot{t}^{-\sum_{i=0}^{r-1} \sigma^i(\alpha^\vee)} L^+\mathcal{G},$$

which lies in the image of $X^M(\mu_{x'}, b)_{K_M}$.

We now consider the case $r \in [h+1, 2h-1]$. In this case, $\#\mathcal{O}_\alpha = 3h$ and each connected component of the Dynkin diagram of G over $\breve{\mathbb{Q}}_p$ is of type D_4. Then either $J = \emptyset$ or $J = \mathcal{O}_\beta$ where β is the unique root in Σ with $\sigma^h(\beta)$ and such that β, α lie in the same connected component of Σ. We consider the following two cases.

Case (i): Either $\langle \beta, \mu_x \rangle = 0$ or $\langle \beta, \alpha^\vee \rangle = 0$. Then, as in [39, Lemma 7.15, Case 2.2], we have $\langle \sigma^j(\alpha), \mu_x \rangle = 0$ and $w_x(\sigma^j(\alpha)) = \sigma^j(\alpha)$ for $j \in [1, r-1]$. Then we may define $u : \mathbb{A}^{1,\mathrm{perf}} \to \mathrm{Gr}_{\mathcal{G}}$ by

$$u(z) = g_x u_\alpha(z)^{b_x\sigma} u_\alpha(z) \dots {}^{(b_x\sigma)^{r-1}} u_\alpha(z) L^+\mathcal{G}$$

as above. Then u extends to $g : \mathbb{P}^{1,\mathrm{perf}} \to \mathrm{Gr}_{\mathcal{G}}$ and the same computation shows that g is a curve connecting $P = g(0) \in X^M(\mu_x, b)$ and $P' := g(\infty) = g_x \dot{t}^{-\sum_{i=0}^{r-1} \sigma^i(\alpha^\vee)} L^+\mathcal{G} \in X^M(\mu_{x'}, b)$.

Case (ii): $\langle \beta, \alpha^\vee \rangle = -1$ and $\langle \beta, \mu_x \rangle = 1$. Then by [39, Lemma 7.15, Case 2], upon switching the roles of x and x' if necessary, we may assume that

$$\langle \sigma^r(\beta), \mu_x \rangle = \langle \sigma^{r-h}(\alpha), \mu_x \rangle = \langle \sigma^h(\alpha), \mu_x \rangle = 0.$$

We define $u : \mathbb{A}^{1,\mathrm{perf}} \to \mathrm{Gr}_{\mathcal{G}}$ by

$$u(z) = g_x {}^{(b_x\sigma)^{r-1}} u_\alpha(z)^{(b_x\sigma)^{r-2}} u_\alpha(z) \dots u_\alpha(z) L^+\mathcal{G}$$

Then u extends to $g : \mathbb{P}^{1,\mathrm{perf}} \to \mathrm{Gr}_{\mathcal{G}}$ and we have

$$g(z)^{-1} b\sigma(g(z)) \in L^+\mathcal{G} u_{\sigma^{r-h}(\alpha)}(-c_2 z)(\dot{t}^\lambda u_{\sigma^r(\alpha)+\sigma^r(\beta)}(c_1 z)\dot{t}^{-\lambda}) u_{\sigma^{r-h}(\alpha)}(c_2 z) u_\alpha(-z) b_x L^+\mathcal{G}$$

where $\lambda \in X_*(T)_I$ satisfies $\langle \sigma^r(\alpha) + \sigma^r(\beta), \lambda \rangle \geq 2$. By Lemma A.3.9 (1) we have

$$u_\alpha(-z) b_x \in \bigcup_{\mu' \preccurlyeq \mu} L^+\mathcal{G} \dot{t}^{\mu'} L^+\mathcal{G}$$

and by Lemma A.3.9 (2) we have

$$u_{\sigma^{r-h}(\alpha)}(-c_2 z)(\dot{t}^\lambda u_{\sigma^r(\alpha)+\sigma^r(\beta)}(c_1 z)\dot{t}^{-\lambda}) u_{\sigma^{r-h}(\alpha)}(c_2 z).$$

It follows that

$$g(z)^{-1}b\sigma(g(z)) \in \bigcup_{\mu' \preccurlyeq \mu} L^+\mathcal{G} t^{\mu'} L^+\mathcal{G}$$

and hence g factors through $X(\mu,b)_K$. A similar calculation to the above shows that g is a curve connecting $P = g(0) \in X^M(\mu_x, b)$ and $P' := g(\infty) \in X^M(\mu_{x'}, b)$ with $\kappa_M(P) - \kappa_M(P') = \sum_{i=0}^{r-1} \sigma^i(\alpha^\vee)$. □

Proof of Proposition A.1.7. This follows by combining Lemma A.3.4 and Lemma A.3.11. □

A.4. **Uniformisation in the case of very special level structure.**

A.4.1. We will use Theorem A.1.3 to give a description of the isogeny classes in $\mathscr{S}_{\mathrm{K}}(G,X)$. We assume that $p > 2$, $p \nmid |\pi_1(G_{\mathrm{der}})|$ and that $G_{\mathbb{Q}_p}$ is quasi-split and splits over a tamely ramified extension of $\mathbb{Q}_p$. We now follow the notation in Section 3.1, so (G,X) is a Shimura datum of Hodge type. We let $U = U^pU_p$ where $U_p \subset G(\mathbb{A}_f^p)$ is a compact open subgroup and U_p is a very special connected parahoric subgroup of $G(\mathbb{Q}_p)$; we write $\mathcal{G}$ for the corresponding parahoric group scheme.

A.4.2. Recall that for $x \in \mathrm{Sh}_{G,K,U^p}(\overline{\mathbb{F}}_p)$ there is an attached abelian variety $\mathcal{A}_x$ with contravariant Dieudonné module $\mathbb{D}_x$ equipped with tensors $s_{\alpha,0,x}$. Moreover for all $\ell \neq p$ the ℓ-adic Tate module $T_\ell\mathcal{A}_x$ is equipped with tensors $s_{\alpha,\ell,x} \in T_\ell\mathcal{A}_x^\otimes$. By [55, Section 5.6], there is an isomorphism

$$V_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \breve{\mathbb{Z}}_p \cong \mathbb{D}_x,$$

taking s_α to $s_{\alpha,0,x}$. Under this identification, the Frobenius on $\mathbb{D}_x$ is of the form $\varphi = b\sigma$ for some $b \in G(\breve{\mathbb{Q}}_p)$; then b is well-defined up to σ-conjugation by $\mathcal{G}_K(\breve{\mathbb{Z}}_p)$.

We let $\mu' \in X_*(T)_I^+$ denote the image of a dominant representative of the conjugacy class $\{\mu_h^{-1}\}$, and we define $\mu = \sigma(\mu')$ as in Section 3.1.6. Then by the argument in [55, Section 5.6], we have

$$b \in \mathcal{G}_K(\breve{\mathbb{Z}}_p)\dot{w}\mathcal{G}_K(\breve{\mathbb{Z}}_p)$$

for some $w \in \mathrm{Adm}(\mu)_K$; it follows that $1 \in X(\mu,b)_K(\overline{\mathbb{F}}_p)$. As in [55, Section 6.7], there is a natural map

$$i'_x : X(\mu,b)_K(\overline{\mathbb{F}}_p) \to \mathscr{S}_{U_V}(G_V, \mathcal{H}_V)(\overline{\mathbb{F}}_p)$$

defined using Dieudonné theory, which sends 1 to the image of x under $\mathrm{Sh}_{G,K,U^p}(\overline{\mathbb{F}}_p) = \mathscr{S}_U(G,X)(\overline{\mathbb{F}}_p) \to \mathscr{S}_{U_V}(G_V, \mathcal{H}_V)(\overline{\mathbb{F}}_p)$.

Let r be the residue degree of the extension $E_v/\mathbb{Q}_p$. Then $X(\mu,b)_K$ is equipped with an action Φ given by $\Phi(g) = (b\sigma)^r(g)$.

Proposition A.4.3 (cf. [55, Proposition 6.5])**.** Suppose U_p is a very special connected[15] parahoric subgroup of $G(\mathbb{Q}_p)$. Then there exists a unique map

$$i_x : X(\mu,b)_K(\overline{\mathbb{F}}_p) \to \mathscr{S}_U(G,X)(\overline{\mathbb{F}}_p)$$

lifting i'_x such that $s_{\alpha,0,i_x(g)} = s_{\alpha,0,x}$ and $\Phi \circ i_x = i_x \circ \Phi$, where Φ acts on $\mathscr{S}_U(G,X)(\overline{\mathbb{F}}_p)$ via the geometric r-Frobenius.

Proof. For notational simplicity, we write $X(\mu,b)_K$ for $X(\mu,b)_K(\overline{\mathbb{F}}_p)$. The uniqueness and compatibility with Φ is proved in the same way as [55, Proposition 6.5]. We may thus define $X(\mu,b)_K^\circ \subset X(\mu,b)_K$ as the maximal subset which admits such a lifting. We therefore need to show that $X(\mu,b)_K^\circ = X(\mu,b)_K$. To do this, we follow the strategy of [55, Proposition 6.5].

[15] See Section 2.2.1 for the definition of connected parahoric.

Arguing as in [55, Lemma 6.10], we have that $X(\mu,b)^\circ_K$ is (the set of $\overline{\mathbb{F}}_p$-points of) a union of connected components. Note that the key input [55, Proposition 6.9] needed for this can be proved verbatim in our setting.

It therefore suffices to show that the map

$$X(\mu,b)^\circ_K \to \pi_0(X(\mu,b)_K)$$

is a surjection. Let $M \subset G_{\mathbb{Q}_p}$ be the standard Levi subgroup given by the centraliser of the Newton cocharacter $\overline{\nu}_b$. By Theorem A.1.3, there exists $\lambda \in I_{\mu,b,M}$ and an element

$$g \in X(\mu,b)^\circ_K \cap X^M(\lambda,b)_M.$$

Upon replacing x by $i_x(g)$ and using the diagram [55, Equation (6.7)], we may assume $b \in M(\breve{\mathbb{Q}}_p)$. Since b is basic in M and using [19, Theorem 4.1], we may further assume that $b = \dot{\tau}_\lambda$ where $\tau_\lambda \in \Omega_M$ corresponds to $\kappa_M(b) \in \pi_1(M)_I$.

Arguing as in [55, Lemma 6.11], we find that [55, Assumption 5.12] is satisfied, in other words, the Hodge filtration on $\mathbb{D}_x \otimes \overline{\mathbb{F}}_p$ lifts to a filtration on $\mathbb{D}_x \otimes \mathcal{O}_K$ for some $K/\breve{\mathbb{Q}}_p$ finite which is induced by an M-valued cocharacter μ_y. We may therefore let $\tilde{\mathscr{G}}/\mathcal{O}_{\breve{K}'}$ be an (M,μ_y)-adapted lifting of $\mathscr{G}$ (cf. [55, Definition 4.6]) which corresponds to a point $\tilde{x} \in \mathscr{S}_U(G,X)(\mathcal{O}_{\breve{K}'})$. The construction in [55, Proposition 6.5] gives us a map

$$\iota : M(\mathbb{Q}_p)/\mathcal{M}(\mathbb{Z}_p) \to X^M(\lambda,b)_{K_M}, \quad g \mapsto g_0$$

which induces a surjection

$$M(\mathbb{Q}_p)/\mathcal{M}(\mathbb{Z}_p) \to \pi_0(X^M(\lambda,b)_{K_M})$$

by [55, Proposition 5.19]. Moreover, the image of ι lands in $X(\mu,b)^\circ_K$. Therefore by Theorem A.1.3, $X(\mu,b)^\circ_K$ intersects every connected component of $X^M(\lambda,b)_{K_M}$, and hence $X(\mu,b)^\circ_K \to \pi_0(X(\mu,b)_K)$ is a surjection as desired.

□

A.4.4. Proposition A.4.3 implies that [55, Assumption 6.17] is satisfied, hence we obtain Theorem A.4.5 below.

Theorem A.4.5. *Let $p > 2$ and (G,X) a Shimura datum of Hodge type with $G_{\mathbb{Q}_p}$ tamely ramified and quasi split. We assume that $p \nmid |\pi_1(G_{\mathrm{der}})|$ and that U_p is a very special connected parahoric subgroup of $G(\mathbb{Q}_p)$.*

(1) Let $x \in \mathscr{S}_{U_p}(G,X)(\overline{\mathbb{F}}_p)$ and $b \in G(\breve{\mathbb{Q}}_p)$ the associated element. Then there is a $G(\mathbb{A}_f^p)$-equivariant bijection (where $\mathscr{I}_x \subset \mathscr{S}_{U_p}(G,X)(\overline{\mathbb{F}}_p)$ is the isogeny class of x)

$$I_x(\mathbb{Q})\backslash X(\mu,b)_K(\overline{\mathbb{F}}_p) \times G(\mathbb{A}_f^p) \to \mathscr{I}_x.$$

(2) Each isogeny class of $\mathscr{S}_{U_p}(G,X)(\overline{\mathbb{F}}_p)$ contains a point x which is the reduction of a special point on $\mathbf{Sh}_{U_p}(G,X)$. This confirms [28, Conjecture 1].

References

[1] Fabrizio Andreatta, *On two mod p period maps: Ekedahl–Oort and fine Deligne–Lusztig stratifications*, arXiv e-prints (March 2021), 2103.12361.

[2] Bhargav Bhatt and Peter Scholze, *Projectivity of the Witt vector affine Grassmannian*, Invent. Math. **209** (2017), no. 2, 329–423.

[3] Mikhail Borovoi, *Abelian Galois cohomology of reductive groups*, Mem. Amer. Math. Soc. **132** (1998), no. 626, viii+50.

[4] ———, *Quotienting $G(\mathbb{Q})_+$ by $G^{sc}(\mathbb{Q})$ and inner forms*, 2020. `https://mathoverflow.net/q/355287` (version: 2020-03-20).

[5] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée*, Inst. Hautes Études Sci. Publ. Math. **60** (1984), 197–376.
[6] Miaofen Chen, Mark Kisin, and Eva Viehmann, *Connected components of affine Deligne-Lusztig varieties in mixed characteristic*, Compos. Math. **151** (2015), no. 9, 1697–1762.
[7] Pierre Deligne, *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, 1979, pp. 247–289.
[8] Torsten Ekedahl and Gerard van der Geer, *Cycle classes of the E-O stratification on the moduli of abelian varieties*, Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I, 2009, pp. 567–636.
[9] Laurent Fargues and Peter Scholze, *Geometrization of the local Langlands correspondence*, arXiv e-prints (February 2021), 2102.13459.
[10] Ian Gleason, Dong Gyu Lim, and Yujie Xu, *The connected components of affine Deligne–Lusztig varieties*, arXiv e-prints (August 2022), 2208.07195.
[11] Ian Gleason and João Lourenço, *On the connectedness of p-adic period domains*, arXiv e-prints (October 2022), 2210.08625.
[12] Thomas J. Haines, *The stable Bernstein center and test functions for Shimura varieties*, Automorphic forms and Galois representations. Vol. 2, 2014, pp. 118–186.
[13] Thomas J. Haines and Timo Richarz, *The test function conjecture for parahoric local models*, J. Amer. Math. Soc. **34** (2021), no. 1, 135–218.
[14] Paul Hamacher and Wansu Kim, *l-adic étale cohomology of Shimura varieties of Hodge type with non-trivial coefficients*, Math. Ann. **375** (2019), no. 3-4, 973–1044.
[15] Paul Hamacher and Eva Viehmann, *Irreducible components of minuscule affine Deligne-Lusztig varieties*, Algebra Number Theory **12** (2018), no. 7, 1611–1634.
[16] X. He and M. Rapoport, *Stratifications in the reduction of Shimura varieties*, Manuscripta Math. **152** (2017), no. 3-4, 317–343.
[17] Xuhua He, *Geometric and homological properties of affine Deligne-Lusztig varieties*, Ann. of Math. (2) **179** (2014), no. 1, 367–404.
[18] ———, *Kottwitz-Rapoport conjecture on unions of affine Deligne-Lusztig varieties*, Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), no. 5, 1125–1141.
[19] Xuhua He and Rong Zhou, *On the connected components of affine Deligne–Lusztig varieties*, Duke Math. J. **169** (2020), no. 14, 2697–2765.
[20] Xuhua He, Rong Zhou, and Yihang Zhu, *Stabilizers of irreducible components of affine Deligne–Lusztig varieties*, arXiv e-prints (September 2021), arXiv:2109.02594.
[21] Jens Hesse, *Central leaves on Shimura varieties with parahoric reduction*, arXiv e-prints (March 2020), 2003.03175.
[22] Manuel Hoff, *Parahoric $(\mathcal{G}, \mu)$-displays and the EKOR-stratification on Shimura varieties of Hodge type* (2023). PhD Thesis in preparation.
[23] Pol van Hoften, *On the ordinary Hecke orbit conjecture*, arXiv e-prints (December 2021), available at `2112.12422`.
[24] Pol van Hoften and Luciena Xiao Xiao, *Monodromy and Irreducibility of Igusa Varieties*, arXiv e-prints (February 2021), 2102.09870.
[25] M. Kisin and G. Pappas, *Integral models of Shimura varieties with parahoric level structure*, Publ. Math. Inst. Hautes Études Sci. **128** (2018), 121–218.
[26] Mark Kisin, *Integral models for Shimura varieties of abelian type*, J. Amer. Math. Soc. **23** (2010), no. 4, 967–1012.
[27] ———, *mod p points on Shimura varieties of abelian type*, J. Amer. Math. Soc. **30** (2017), no. 3, 819–914.
[28] Mark Kisin, Keerthi Madapusi Pera, and Sug Woo Shin, *Honda-Tate theory for Shimura varieties*, Duke Math. J. **171** (2022), no. 7, 1559–1614. MR4484214
[29] Robert E. Kottwitz, *Isocrystals with additional structure. II*, Compositio Math. **109** (1997), no. 3, 255–339.
[30] Arno Kret and Sug Woo Shin, *H^0 of Igusa varieties via automorphic forms*, arXiv e-prints (February 2021), arXiv:2102.10690, available at `2102.10690`.
[31] R. P. Langlands, *Some contemporary problems with origins in the Jugendtraum*, Mathematical developments arising from Hilbert problems (Proc. Sympos. Pure Math., Vol. XXVIII, Northern Illinois Univ., De Kalb, Ill., 1974), 1976, pp. 401–418.
[32] ———, *Shimura varieties and the Selberg trace formula*, Canadian J. Math. **29** (1977), no. 6, 1292–1299.
[33] R. P. Langlands and M. Rapoport, *Shimuravarietäten und Gerben*, J. Reine Angew. Math. **378** (1987), 113–220.
[34] Eike Lau, *Dieudonné theory over semiperfect rings and perfectoid rings*, Compos. Math. **154** (2018), no. 9, 1974–2004. MR3867290

[35] George Lusztig, *Singularities, character formulas, and a q-analog of weight multiplicities*, Analysis and topology on singular spaces, II, III (Luminy, 1981), 1983, pp. 208–229.
[36] Keerthi Madapusi Pera, *Toroidal compactifications of integral models of Shimura varieties of Hodge type*, Ann. Sci. Éc. Norm. Supér. (4) **52** (2019), no. 2, 393–514.
[37] J. S. Milne, *Introduction to Shimura varieties*, Harmonic analysis, the trace formula, and Shimura varieties, 2005, pp. 265–378.
[38] James S. Milne, *The points on a Shimura variety modulo a prime of good reduction*, The zeta functions of Picard modular surfaces, 1992, pp. 151–253.
[39] Sian Nie, *Connected components of closed affine Deligne-Lusztig varieties in affine Grassmannians*, Amer. J. Math. **140** (2018), no. 5, 1357–1397.
[40] Sian Nie, *Connectedness of affine Deligne-Lusztig varieties for unramified groups*, arXiv e-prints (July 2021), 2107.05205.
[41] G. Pappas and M. Rapoport, *Twisted loop groups and their affine flag varieties*, Adv. Math. **219** (2008), no. 1, 118–198. With an appendix by T. Haines and Rapoport.
[42] Georgios Pappas, *On integral models of Shimura varieties*, Mathematische Annalen (2022Apr).
[43] Georgios Pappas and Michael Rapoport, *p-adic shtukas and the theory of global and local Shimura varieties*, arXiv e-prints (June 2021), 2106.08270.
[44] Vladimir Platonov and Andrei Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics, vol. 139, Academic Press, Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen.
[45] M. Rapoport and M. Richartz, *On the classification and specialization of F-isocrystals with additional structure*, Compositio Math. **103** (1996), no. 2, 153–181.
[46] M. Rapoport and Th. Zink, *Period spaces for p-divisible groups*, Annals of Mathematics Studies, vol. 141, Princeton University Press, Princeton, NJ, 1996.
[47] Michael Rapoport, *A guide to the reduction modulo p of Shimura varieties*, 2005, pp. 271–318. Automorphic forms. I.
[48] Michael Rapoport and Eva Viehmann, *Towards a theory of local Shimura varieties*, Münster J. Math. **7** (2014), no. 1, 273–326.
[49] Harry Reimann, *The semi-simple zeta function of quaternionic Shimura varieties*, Lecture Notes in Mathematics, vol. 1657, Springer-Verlag, Berlin, 1997.
[50] Xu Shen, Chia-Fu Yu, and Chao Zhang, *EKOR strata for Shimura varieties with parahoric level structure*, Duke Math. J. **170** (2021), no. 14, 3111–3236.
[51] The Stacks project authors, *The stacks project*, 2023.
[52] J. Tits, *Reductive groups over local fields*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, 1979, pp. 29–69.
[53] Torsten Wedhorn and Paul Ziegler, *Tautological rings of Shimura varieties and cycle classes of Ekedahl-Oort strata*, arXiv e-prints (2018), 1811.04843.
[54] Liang Xiao and Xinwen Zhu, *Cycles on Shimura varieties via geometric Satake*, arXiv e-prints (2017), 1707.05700.
[55] Rong Zhou, *Mod p isogeny classes on Shimura varieties with parahoric level structure*, Duke Math. J. **169** (2020), no. 15, 2937–3031.
[56] Rong Zhou and Yihang Zhu, *Twisted orbital integrals and irreducible components of affine Deligne-Lusztig varieties*, Camb. J. Math. **8** (2020), no. 1, 149–241.
[57] Xinwen Zhu, *Affine Grassmannians and the geometric Satake in mixed characteristic*, Ann. of Math. (2) **185** (2017), no. 2, 403–492.

Stanford mathematics department, 450 Jane Stanford way, Building 380, Stanford, CA 94305, USA
Email address: pol.van.hoften@stanford.edu